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Quasivarieties with definable relative principal subcongruences

Abstract. For quasivarieties of algebras, we consider the property of having definable relative principal subcongruences, a generalization of the concepts of definable relative principal congruences and definable principal subcongruences. We prove that a quasivariety of algebras with definable relative principal subcongruences has a finite quasi-equational basis if and only if the class of its relative (finitely) subdirectly irreducible algebras is strictly elementary. Since a finitely generated relatively congruence-distributive quasivariety has definable relative principal subcongruences, we get a new proof of the result due to D. Pigozzi: a finitely generated relatively congruence-distributive quasivariety has a finite quasi-equational basis.

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1. Introduction

In [2] K. A. Baker and J. Wang introduced the notion of *definable principal subcongruences* (DPSC). With the aid of it, they provided a new proof of celebrated Baker's theorem [1]: each finitely generated congruence-distributive variety has finite equational basis. (All algebras and classes of algebras considered in this paper are assumed to be in a finite language.) The novelty of this proof lies not only in its shortness and exceptional elegance, but also in the fact that it does not relay on the existence of Jónsson's terms [9]. Therefore it is plausible that the DPSC technique may be carried over to a more general setting. The aim of this article is to demonstrate that it is the case indeed. We introduce the notion of *definable relative principal subcongruences* (DRPSC) and employ it to provide a new proof of the following generalization of Baker's theorem obtained by D. Pigozzi [19].

Theorem 1. *A finitely generated relatively congruence-distributive quasivariety has a finite quasi-equational basis.*

Our main contribution is the following theorem.

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THEOREM 2. *Let $\mathcal{Q}$ be a quasivariety with DRPSC and let $\mathcal{Q}_{SI}$ ($\mathcal{Q}_{FSI}$) be the class of (finitely) $\mathcal{Q}$-subdirectly irreducible algebras. Then the following conditions are equivalent*

❶ *$\mathcal{Q}$ has a finite quasi-equational basis;*

❷ *$\mathcal{Q}_{SI}$ is strictly elementary;*

❸ *$\mathcal{Q}_{FSI}$ is strictly elementary.*

Several results preceded our Theorem 2. R. McKenzie [13] proved that a variety $\mathcal{V}$ with definable principal congruences (DPC) which is residually less than n, for some finite n, has finite equational basis. This was generalized by B. Jónsson and K. A. Baker [10] who showed that a variety $\mathcal{V}$ with DPC and $\mathcal{V}_{SI}$ ($\mathcal{V}_{FSI}$) strictly elementary has finite equational basis. The extension of this theorem to the quasivariety case by J. Czelakowski, W. Dziobiak [5] and the first author [16] followed. Finally, K. A. Baker and J. Wang proved [2] that a variety $\mathcal{V}$ with DPSC and $\mathcal{V}_{SI}$ ($\mathcal{V}_{FSI}$) strictly elementary has finite equational basis.

The new proof of Baker's theorem presented in [2] is based on the above mentioned result of K. A. Baker and J. Wang, and the fact that each finitely generated congruence-distributive variety has DPSC [2, Theorem 2]. In order to obtain a proof of Pigozzi's theorem one can easily carry over the latter fact to the quasivariety case (Proposition 7) and use our Theorem 2.

Pigozzi's theorem was already reproved by W. Dziobiak in [6] and by M. Maróti, R. McKenzie in [12]. But, according to our knowledge, the proof presented here is the shortest one, as is the proof of Baker's theorem given in [2]. However, both these proofs are non-constructive.

The results more general than Pigozzi's theorem are known nowadays. The first author of this article proved in [17] that a locally finite relatively congruence-distributive quasivariety $\mathcal{Q}$ has a finite quasi-equational basis provided the class $\mathcal{Q}_{FSI}$ is strictly elementary. W. Dziobiak, M. Maróti, R. McKenzie and the first author proved [7] that each finitely generated relatively congruence-meet-semidistributive quasivariety has a finite quasi-equational basis. It is a quasivariety version of Willard's theorem [20]. Finally, K. Pałasińska proved in [18] that a finitely generated protoalgebraic filter-distributive strict universal Horn class is finitely axiomatizable.

2. Toolbox

We briefly recall basic facts from quasivariety theory we will need. A standard book about quasivarieties is [8] and about universal algebra are [4, 14]. One may also consult the classical position [11].

We assume that all considered algebras are in the same finite language. A *quasivariety* $\mathcal{Q}$ is a class of algebras defined by a set Σ of quasi-identities (strict universal Horn sentences). The set Σ is called a *quasi-equational basis* of $\mathcal{Q}$. Recall that a class is *strictly elementary*, or *finitely axiomatizable*, if it may be defined by a single sentence. Notice that a quasivariety $\mathcal{Q}$ has a finite quasi-equational basis if and only if $\mathcal{Q}$ is strictly elementary. The smallest quasivariety $\mathcal{Q}$ containing a given class $\mathcal{G}$ is $\mathbf{SPP_U}(\mathcal{G})$, where $\mathbf{P}$ denotes the product class operator, $\mathbf{P_U}$ denotes the ultraproduct class operator, and $\mathbf{S}$ denotes the subalgebra class operator [8, Corollary 2.3.4]. If $\mathcal{G}$ is a finite family of finite algebras, then $\mathcal{Q} = \mathbf{SP}(\mathcal{G})$, and every such $\mathcal{Q}$ is called *finitely generated.*

Let $\mathcal{Q}$ be a quasivariety. A congruence α on an algebra A is called a $\mathcal{Q}$*-congruence* provided $A/\alpha \in \mathcal{Q}$. Note that $A \in \mathcal{Q}$ if and only if the equality relation 0_A on A is a $\mathcal{Q}$-congruence. The set $\mathrm{Con}_{\mathcal{Q}}(A)$ of all $\mathcal{Q}$-congruences of A forms an algebraic lattice which is a meet-subsemilattice of $\mathrm{Con}(A)$ [8, Corollary 1.4.11]. We use the symbol $\vee^{\mathcal{Q}}$ for the lattice join in $\mathrm{Con}_{\mathcal{Q}}(A)$.

A nontrivial algebra $S \in \mathcal{Q}$ is $\mathcal{Q}$*-subdirectly irreducible* if 0_S is a completely meet irreducible in $\mathrm{Con}_{\mathcal{Q}}(A)$, and S is *finitely* $\mathcal{Q}$*-subdirectly irreducible* if 0_S is a meet irreducible in $\mathrm{Con}_{\mathcal{Q}}(A)$. Let us denote the class of all (finitely) $\mathcal{Q}$-subdirectly irreducible algebras by $\mathcal{Q}_{SI}$ ($\mathcal{Q}_{FSI}$). Note that if $\mathcal{Q} = \mathbf{SP}(\mathcal{G})$, then [8, Proposition 3.1.6]

$$\mathcal{Q}_{SI} \subseteq \mathbf{S}(\mathcal{G}). \tag{1}$$

In an algebraic lattice each element is a meet of completely meet-irreducible elements. For $A \in \mathcal{Q}$ the lattice $\mathrm{Con}_{\mathcal{Q}}(A)$ is algebraic. Thus we obtain [8, Theorem 3.1.1]

$$\mathcal{Q} = \mathbf{P_S}(\mathcal{Q}_{SI}), \tag{2}$$

where $\mathbf{P_S}$ denotes the subdirect product class operator.

Let A be an algebra and $H \subseteq A^2$. Then there exists the least $\mathcal{Q}$-congruence containing H and we denote it by $\theta^A_{\mathcal{Q}}(H)$. Note that $\theta^A_{\mathcal{Q}}(-)$ is the algebraic closure operator associated to the lattice $\mathrm{Con}_{\mathcal{Q}}(A)$. If H consists of one element (a, b) we simply write $\theta^A_{\mathcal{Q}}(a, b)$ and call such congruence a *principal* $\mathcal{Q}$*-congruence* or a *relative principal congruence*. Notice that for $\alpha \in \mathrm{Con}_{\mathcal{Q}}(A)$ and $a, b, c, d \in A$ we have the following useful equivalence

$$(c, d) \in \alpha \vee^{\mathcal{Q}} \theta^A_{\mathcal{Q}}(a, b) \quad \Leftrightarrow \quad (c/\alpha, d/\alpha) \in \theta^{A/\alpha}_{\mathcal{Q}}(a/\alpha, b/\alpha). \tag{3}$$

3. DRPSC

An existential positive formula $\Gamma(x, y, u, v)$ is an *$\mathcal{Q}$-congruence formula* if

$$\mathcal{Q} \models (\forall x, u, v)[\Gamma(u, v, x, x) \to u \approx v] \tag{4}$$

[8, Subsection 1.4.4]. For example $ux \approx vy \vee xu \approx yv$ is a congruence formula for the quasivariety of cancellative groupoids. The importance of $\mathcal{Q}$-congruence formulas follows from the following fact [8, Proposition 1.4.13].

PROPOSITION 3. *Let $\mathcal{Q}$ be a quasivariety and $a, b, c, d \in A \in \mathcal{Q}$. Then $(c, d) \in \theta^A_{\mathcal{Q}}(a, b)$ if and only if there exists a $\mathcal{Q}$-congruence formula Γ such that $A \models \Gamma(c, d, a, b)$.*

Let $a, b \in A \in \mathcal{Q}$. We say that a $\mathcal{Q}$-congruence formula Γ defines $\theta^A_{\mathcal{Q}}(a, b)$ if

$$\theta^A_{\mathcal{Q}}(a, b) = \{(c, d) \in A^2 \mid A \models \Gamma(c, d, a, b)\}.$$

We say that $\mathcal{Q}$ has *definable relative principal congruences* (DRPC) if there exists a $\mathcal{Q}$-congruence formula defining all principal $\mathcal{Q}$-congruences in $\mathcal{Q}$. In the variety case this reduces to having definable principal congruences (DPC). A quasivariety $\mathcal{Q}$ has *definable relative principal subcongruences* (DRPSC) if there exists a $\mathcal{Q}$ congruence formula Γ such that for each pair a, b of distinct elements of $A \in \mathcal{Q}$ there exists a pair $c, d \in A$ of distinct elements such that

$$A \models \Gamma(c, d, a, b) \quad \text{and} \quad \theta^A_{\mathcal{Q}}(c, d) = \{(e, f) \mid A \models \Gamma(e, f, c, d)\}.$$

Similarly as above, this reduces to having definable principal subcongruences (DPSC) in the variety case. The property of having DRPSC is more common than of having DRPC (Proposition 7 and Remark 8). On the other hand, it still allows to prove the finite axiomability theorem (Theorem 2).

4. Proof of Theorem 2

We say that a quasivariety $\mathcal{Q} \subseteq \mathcal{U}$ is *finitely axiomatizable relative to $\mathcal{U}$* if there exists a finitely axiomatizable quasivariety $\mathcal{R}$ such that $\mathcal{Q} = \mathcal{R} \cap \mathcal{U}$. Equivalently, there is a finite set Σ of quasi-identities such that, for every algebra $A \in \mathcal{U}$, $A \in \mathcal{Q}$ if and only if $A \models \Sigma$. Our first step is to show that each quasivariety $\mathcal{Q}$ with DRPSC is finitely axiomatizable relative to the variety it generates. This means that $\mathcal{Q}$ has a quasi-equational basis of the form $\Sigma \cup I$, where Σ is finite and I is a set of identities (Lemma 5). For this we need the following lemma.

Lemma 4 (cf. [16]). *Let $\mathcal{Q}$ be a subquasivariety of a quasivariety $\mathcal{R}$. Assume that $\mathcal{Q}$ and $\mathcal{R}$ generate the same variety $\mathcal{V}$. If $\theta^A_{\mathcal{Q}}(a,b) = \theta^A_{\mathcal{R}}(a,b)$ for all $a,b \in A \in \mathcal{Q}$, then $\mathcal{Q} = \mathcal{R}$.*

Proof. Let $A \in \mathcal{Q}$. Note first that if $\alpha = \theta^A_{\mathcal{R}}(\{(a_1,b_1),\dots,(a_n,b_n)\})$ is a finitely generated $\mathcal{R}$-congruence of A, then α is a $\mathcal{Q}$-congruence. Indeed, if $n = 0$, then $\alpha = 0_A$ is a trivial congruence. Let $n > 0$ and assume that $\beta = \theta^A_{\mathcal{R}}(\{(a_1,b_1),\dots,(a_{n-1},b_{n-1})\})$ is a $\mathcal{Q}$-congruence. Then by (3) and the assumption

$$\begin{aligned}\alpha &= \beta \vee^{\mathcal{R}} \theta^A_{\mathcal{R}}(a_n,b_n)\\ &= \{(c,d) \in A^2 \mid (c/\beta, d/\beta) \in \theta^{A/\beta}_{\mathcal{R}}(a_n/\beta, b_n/\beta)\}\\ &= \{(c,d) \in A^2 \mid (c/\beta, d/\beta) \in \theta^{A/\beta}_{\mathcal{Q}}(a_n/\beta, b_n/\beta)\}\\ &= \beta \vee^{\mathcal{Q}} \theta^A_{\mathcal{Q}}(a_n,b_n)\end{aligned}$$

is also a $\mathcal{Q}$-congruence. Thus, for every $A \in \mathcal{Q}$ and every finite subset H of A^2 we have $\theta^A_{\mathcal{Q}}(H) = \theta^A_{\mathcal{R}}(H)$.

Now let $C \in \mathcal{R}$. Since $\mathcal{R}$ generates $\mathcal{V}$, there exists a free algebra F in $\mathcal{V}$ and an $\mathcal{R}$-congruence γ on F such that C is isomorphic to F/γ. Since $\mathcal{Q}$ generates $\mathcal{V}$, $F \in \mathcal{Q}$. Obviously, $\gamma = \theta^F_{\mathcal{R}}(\gamma) \subseteq \theta^F_{\mathcal{Q}}(\gamma)$. Since $\theta^F_{\mathcal{Q}}(-)$ is an algebraic closure operator on 2^{F^2}, it follows from what we have established above that $\theta^F_{\mathcal{Q}}(\gamma) \subseteq \theta^F_{\mathcal{R}}(\gamma)$. Hence $\gamma = \theta^F_{\mathcal{Q}}(\gamma)$, proving that γ is a $\mathcal{Q}$-congruence and $C \in \mathcal{Q}$. ∎

Lemma 5. *Let $\mathcal{Q}$ be a quasivariety with DRPSC. Then $\mathcal{Q}$ is finitely axiomatizable relative to the variety $\mathcal{V}$ it generates.*

Proof. Let Γ be a $\mathcal{Q}$-congruence formula witnessing DRPSC for $\mathcal{Q}$. By the condition (4) and compactness theorem there is a finitely based quasivariety $\mathcal{R}$ containing $\mathcal{Q}$ such that Γ is also an $\mathcal{R}$-congruence formula. We will show that $\mathcal{Q} = \mathcal{V} \cap \mathcal{R}$. Let $A \in \mathcal{Q}$. By Lemma 4, it is enough to show that $\theta^A_{\mathcal{Q}}(a,b) = \theta^A_{\mathcal{V}\cap\mathcal{R}}(a,b)$ for all $a,b \in A$. The inclusion $\theta^A_{\mathcal{R}\cap\mathcal{V}}(a,b) \subseteq \theta^A_{\mathcal{Q}}(a,b)$ is obvious. In order to prove the inverse inclusion we construct a sequence $(\alpha_\kappa)_{\kappa<\varrho}$, where ϱ is an ordinal, of congruences with the following properties.

- $|\operatorname{Con}_{\mathcal{Q}}(A)| < |\{\kappa \mid \kappa < \varrho\}|$;
- if $\lambda \leqslant \kappa < \varrho$, then $\alpha_\lambda \subseteq \alpha_\kappa$;
- for $\kappa < \varrho$, α_κ is a $\mathcal{Q}$-congruence;
- for $\kappa < \varrho$, $\alpha_\kappa \subseteq \theta^A_{\mathcal{R}}(a,b)$;
- for $\kappa < \varrho$, $\alpha_{\kappa+1} = \alpha_\kappa$ implies $\alpha_\kappa = \theta^A_{\mathcal{Q}}(a,b)$.

Note that if $\bigcup \alpha_\kappa \neq \theta^A_{\mathcal{Q}}(a,b)$, then the congruences α_κ form a chain in $\mathrm{Con}_{\mathcal{Q}}(A)$ of cardinality greater than $|\operatorname{Con}_{\mathcal{Q}}(A)|$, what is impossible. Thus

$$\theta^A_{\mathcal{Q}}(a,b) = \bigcup \alpha_\kappa \subseteq \theta^A_{\mathcal{R}}(a,b) = \theta^A_{\mathcal{R}\cap\mathcal{V}}(a,b).$$

We construct $(\alpha_\kappa)_{\kappa<\varrho}$ recursively. First put $\alpha_0 = 0_A$. Now let $\kappa = \lambda + 1 < \varrho$ be a successor ordinal and assume that α_λ is already defined. If $(a,b) \in \alpha_\lambda$, let $\alpha_\kappa = \alpha_\lambda$. Otherwise, because $A/\alpha_\lambda \in \mathcal{Q}$ and Γ witnesses DRPSC for $\mathcal{Q}$, there exists a pair $(c,d) \notin \alpha_\lambda$ of elements of A such that

$$A/\alpha_\lambda \models \Gamma(c/\alpha_\lambda, d/\alpha_\lambda, a/\alpha_\lambda, b/\alpha_\lambda) \tag{5}$$

and

$$\theta^{A/\alpha_\lambda}_{\mathcal{Q}}(c/\alpha_\lambda, d/\alpha_\lambda) = \{(e/\alpha_\lambda, f/\alpha_\lambda) \mid A/\alpha_\lambda \models \Gamma(e/\alpha_\lambda, f/\alpha_\lambda, c/\alpha_\lambda, d/\alpha_\lambda)\}. \tag{6}$$

We define

$$\alpha_\kappa = \alpha_\lambda \vee^{\mathcal{Q}} \theta^A_{\mathcal{Q}}(c,d).$$

We claim that $\alpha_\kappa = \alpha_\lambda \vee^{\mathcal{R}} \theta^A_{\mathcal{R}}(c,d)$.
Indeed, by (3)

$$\alpha_\kappa = \{(e,f) \in A^2 \mid (e/\alpha_\lambda, f/\alpha_\lambda) \in \theta^{A/\alpha_\lambda}_{\mathcal{Q}}(c/\alpha_\lambda, d/\alpha_\lambda)\},$$

and by (6)

$$\alpha_\kappa = \{(e,f) \in A^2 \mid A/\alpha_\lambda \models \Gamma(e/\alpha_\lambda, f/\alpha_\lambda, c/\alpha_\lambda, d/\alpha_\lambda)\}.$$

But Γ is an $\mathcal{R}$-congruence formula and $A/\alpha_\lambda \in \mathcal{R}$, hence by Proposition 3 and (3)

$$\alpha_\kappa \subseteq \{(e,f) \in A^2 \mid (e/\alpha_\lambda, f/\alpha_\lambda) \in \theta^{A/\alpha_\lambda}_{\mathcal{R}}(c/\alpha_\lambda, d/\alpha_\lambda)\} = \alpha_\lambda \vee^{\mathcal{R}} \theta^A_{\mathcal{R}}(c,d).$$

This proves the claim.

Similarly by Proposition 3 and (3), (5) yields $(c,d) \in \alpha_\lambda \vee^{\mathcal{R}} \theta^A_{\mathcal{R}}(a,b)$. This together with $\alpha_\lambda \subseteq \theta^A_{\mathcal{R}}(a,b)$ imply that $(c,d) \in \theta^A_{\mathcal{R}}(a,b)$. Hence, by the claim, $\alpha_\kappa \subseteq \theta^A_{\mathcal{R}}(a,b)$.

It remains to define α_κ when κ is a limit ordinal and α_λ is defined for all ordinals $\lambda < \kappa$. It may be done by putting $\alpha_\kappa = \bigcup_{\lambda<\kappa} \alpha_\lambda$. ■

Proof of Theorem 2.
❶ ⇒ ❷. Let Γ be a $\mathcal{Q}$-congruence formula witnessing DRPSC for $\mathcal{Q}$. Let $\Gamma^2(r,s,x,y)$ be the abbreviation of $(\exists u,v)[\Gamma(u,v,x,y) \wedge \Gamma(r,s,u,v)]$ and put

$$\psi = (\exists r,s)\Big[r \not\approx s \wedge (\forall x,y)\big[(x \not\approx y \to \Gamma^2(r,s,x,y))\big]\Big]. \tag{7}$$

Observe that $S \in \mathcal{Q}$ is $\mathcal{Q}$-subdirectly irreducible if and only if $S \models \psi$. Thus if Λ is a finite quasi-equational basis of $\mathcal{Q}$, then $\Lambda \cup \{\psi\}$ defines $\mathcal{Q}_{SI}$ and this class is strictly elementary.

❷ $\Rightarrow$ ❶. Let I be a set of identities and Σ be a set of quasi-identities such that $\Sigma \cup I$ defines $\mathcal{Q}$. By Lemma 5, we may assume that Σ is a finite. For a quasi-identity

$$\sigma = (\forall \bar{x})\Big[\Big[\bigwedge_{i \leqslant n} p_i(\bar{x}) \approx q_i(\bar{x})\Big] \to p(\bar{x}) \approx q(\bar{x})\Big]$$

let

$$\delta_\sigma(u, v) = (\forall \bar{x})\Big[\Big[\bigwedge_{i \leqslant n} \Gamma(p_i(\bar{x}), q_i(\bar{x}), u, v)\Big] \to \Gamma(p(\bar{x}), q(\bar{x}), u, v)\Big].$$

Put

$$\delta(u, v) = \Gamma(u, v, u, v) \wedge \bigwedge_{\sigma \in \Sigma \cup Eq} \delta_\sigma(u, v),$$

where Eq is the set of axioms for the equality relation. Observe that $A \models \delta(c, d)$ if and only if $\beta = \big\{(e, f) \in A^2 \mid A \models \Gamma(e, f, c, d)\big\}$ is a congruence containing (c, d) and $A/\beta \models \Sigma$. If additionally $A \models I$, $A \models \delta(c, d)$ if and only if β is a $\mathcal{Q}$-congruence containing (c, d). In particular, when $A \in \mathcal{Q}$, $A \models \delta(c, d)$ if and only if Γ defines $\theta^A_{\mathcal{Q}}(c, d)$.

Now let

$$\varphi = (\forall x, y)\Big[x \not\approx y \to \big[(\exists u, v)[u \not\approx v \wedge\ \Gamma(u, v, x, y) \wedge \delta(u, v)]\big]\Big].$$

The consideration given above and the fact that Γ witnesses DRPSC for $\mathcal{Q}$ yield $\mathcal{Q} \models \varphi$. Thus, by the compactness theorem, there is a finite subset I_0 of I such that

$$\Sigma \cup I_0 \models \varphi. \tag{8}$$

It is worth to mention that the quasivariety defined by $\Sigma \cup I_0$ does not need to have DRPSC. This is because Γ does not need to be a congruence formula relative to it.

Since $\mathcal{Q}_{SI}$ is strictly elementary, there is a sentence χ defining $\mathcal{Q}_{SI}$. Whence $\Sigma \cup I \cup \{\psi\} \models \chi$, where ψ is as in (7). By the compactness theorem, there is a finite set $I_1 \subseteq I$ such that

$$\Sigma \cup I_1 \cup \{\psi\} \models \chi. \tag{9}$$

Let $\mathcal{R}$ be the quasivariety defined by $\Sigma \cup I_0 \cup I_1$. We show that $\mathcal{R}_{SI} \subseteq \mathcal{Q}_{SI}$. Then by (2), $\mathcal{R} \subseteq \mathcal{Q}$. And because the inverse inclusion holds, $\mathcal{R} = \mathcal{Q}$. So let $S \in \mathcal{R}_{SI}$ and choose distinct $g, h \in S$ such that (g, h) belongs to each nonzero $\mathcal{R}$-congruence on S. By (8) for arbitrary distinct $a, b \in S$ there are distinct $c, d \in S$ such that $S \models \Gamma(c, d, a, b)$ and $\gamma = \{(e, f) \in S^2 : S \models \Gamma(e, f, c, d)\}$ is an $\mathcal{R}$-congruence containing (c, d). In particular γ is nontrivial, hence $(g, h) \in \gamma$. This yields

$$S \models g \not\approx h \wedge (\forall x, y)\big[(x \not\approx y \to \Gamma^2(g, h, x, y)\big],$$

where Γ^2 is defined as in the proof of ❶ ⇒ ❷. We obtained that $S \models \psi$. By (9), $S \models \chi$, and thus $S \in \mathcal{Q}_{SI}$.

❶ ⇔ ❸. One may proceed similarly. Just instead of ψ from the proof of implication ❶ ⇒ ❷ the sentence

$$(\forall x, y, u, v)\Big[[x \not\approx y \wedge u \not\approx v] \to \Big[(\exists r, s)[r \not\approx s \wedge \Gamma^2(r, s, x, y) \wedge \Gamma^2(r, s, u, v)]\Big]\Big]$$

should be used. ■

5. Relative congruence-distributive quasivarieties

A quasivariety $\mathcal{Q}$ is *relatively congruence-distributive* if the lattice $\mathrm{Con}_{\mathcal{Q}}(A)$ is distributive for all $A \in \mathcal{Q}$. A variety is *congruence-distributive* if it is relatively congruence-distributive. The proof by K. A. Baker and J. Wang [2] that a congruence-distributive finitely generated variety has DPSC carries over easily to the quasivariety case. However, for the sake of completeness, we provide here a detailed exposition for the later case.

LEMMA 6. *Let $\mathcal{Q}$ be a quasivariety and assume that there exists a natural number N with the following property: For each algebra $A \in \mathcal{Q}$ and each pair of distinct elements $a, b \in A$ there exists a pair of distinct elements $c, d \in A$ such that*

- *there exits a subalgebra B of A with at most N elements such that $a, b, c, d \in B$ and $(c, d) \in \theta^B_{\mathcal{Q}}(a, b)$, and*
- *if $(e, f) \in \theta^A_{\mathcal{Q}}(c, d)$, then there exists a subalgebra C of A with at most N elements such that $c, d, e, f \in C$ and $(c, d) \in \theta^C_{\mathcal{Q}}(a, b)$.*

Then $\mathcal{Q}$ has DRPSC.

Proof. Proposition 3 and the fact that a nonempty disjunction of $\mathcal{Q}$-congruence formulas is also a $\mathcal{Q}$-congruence formula yield that for each finite family of finite algebras $\mathcal{G}$ there exists a $\mathcal{Q}$-congruence formula defining principal $\mathcal{Q}$-congruences in $\mathcal{G}$. In particular, there exists a $\mathcal{Q}$-congruence formula defining $\mathcal{Q}$-congruences in all members of $\mathcal{Q}$ of cardinality not greater than N. By the condition in the proposition, Γ witnesses DRPSC for $\mathcal{Q}$. ■

Proposition 7. *A finitely generated relatively congruence-distributive quasivariety $\mathcal{Q}$ has DRPSC.*

Proof. By (1) there exists a finite bound M for the cardinality of the $\mathcal{Q}$-subdirectly irreducible algebras. Let N be the cardinality of the free $\mathcal{Q}$-algebra of rank $M + 2$. Because $\mathcal{Q}$ is finitely generated, N is finite. Moreover, each $\mathcal{Q}$-algebra with at most $M+2$ generators has cardinality not greater than N. We argue that N satisfies the condition of Lemma 6.

Consider an algebra $A \in \mathcal{Q}$ and a pair of distinct elements $a, b \in A$. By (2) there exists a family of $\mathcal{Q}$-congruences $\eta_i \in \mathrm{Con}_{\mathcal{Q}}(A)$, $i \in I$, such that $\bigcap_{i\in I} \eta_i = 0_A$ and all $S_i = A/\eta_i$ are $\mathcal{Q}$-subdirectly irreducible. We choose $j \in I$ so that $|S_j|$ is as large as possible subject to $(a, b) \notin \eta_j$. Let B be a subalgebra of A such that $a, b \in B$ and $|B/\eta_j^B| = |S_j|$, where η_i^D denotes $\eta_i \cap D^2$. This condition means that B contains at least one representative of each η_j class. Because $|S_j| \leqslant M$, we may assume that B is M generated, and hence $|B| \leqslant N$. Since S_j is $\mathcal{Q}$-subdirectly irreducible, $\eta_j^B \neq B^2$ and η_j^B is meet irreducible in $\mathrm{Con}_{\mathcal{Q}}(B)$. Thus, by distributivity of $\mathrm{Con}_{\mathcal{Q}}(B)$, there exists $\alpha \in \mathrm{Con}_{\mathcal{Q}}(B)$ such that $\alpha \nleqslant \eta_j^B$ and for arbitrary $\beta \in \mathrm{Con}_{\mathcal{Q}}(B)$ either $\beta \leqslant \eta_j^B$ or $\alpha \leqslant \beta$. Moreover, α is join-irreducible. Hence there is a pair of distinct element $c, d \in B$ such that $\alpha = \theta_{\mathcal{Q}}^B(c, d)$ and

$$\big(\forall \beta \in \mathrm{Con}_{\mathcal{Q}}(B)\big) \left[\beta \leqslant \eta_j^B \quad \text{xor} \quad \theta_{\mathcal{Q}}^B(c, d) \leqslant \beta\right]. \tag{10}$$

By the choice of j, $(a, b) \notin \eta_j^B$. Thus by (10), $(c, d) \in \theta_{\mathcal{Q}}^B(a, b)$.

Now consider a pair $e, f \in A$ such that $(e, f) \in \theta_{\mathcal{Q}}^A(c, d)$. Let C be a subalgebra of A generated by $B \cup \{e, f\}$. Note that C is $M + 2$ generated, and hence $|C| \leqslant N$. In order to finish the proof we need to verify that $(e, f) \in \theta_{\mathcal{Q}}^C(c, d)$.

The finiteness of C yields that the family $\{\eta_i^C \mid i \in I\}$ is finite. Therefore, by distributivity,

$$\theta_{\mathcal{Q}}^C(c, d) = \theta_{\mathcal{Q}}^C(c, d) \vee^{\mathcal{Q}} 0_C = \theta_{\mathcal{Q}}^C(c, d) \vee^{\mathcal{Q}} \bigcap_{i\in I} \eta_i^C = \bigcap_{i\in I} \big(\theta_{\mathcal{Q}}^C(c, d) \vee^{\mathcal{Q}} \eta_i^C\big)\,.$$

Thus it is enough to show that $(e,f) \in \theta^C_{\mathcal{Q}}(c,d) \vee^{\mathcal{Q}} \eta^C_i$ for all $i \in I$. There are two cases. If $(c,d) \in \eta_i$, then obviously $(e,f) \in \eta^C_i$. Assume that $(c,d) \notin \eta_i$. Then $(a,b) \notin \eta_i$. Moreover, (10) yields $\eta^B_i \leqslant \eta^B_j$. Thus, by the choice of j and B,

$$|S_i| \leqslant |S_j| = |B/\eta^B_j| \leqslant |B/\eta^B_i| \leqslant |C/\eta^C_i| \leqslant |A/\eta_i| = |S_i|.$$

Therefore the mapping $C/\eta^C_i \to A/\eta_i$, given by $x/\eta^C_i \mapsto x/\eta_j$, is an isomorphism. Consequently by (3)

$$\begin{aligned} (e,f) \in \theta^A_{\mathcal{Q}}(c,d) \quad &\Rightarrow \quad (e/\eta_i, f/\eta_i) \in \theta^{A/\eta_i}_{\mathcal{Q}}(c/\eta_i, d/\eta_i) \\ &\Leftrightarrow \quad (e/\eta^C_i, f/\eta^C_i) \in \theta^{C/\eta^C_i}_{\mathcal{Q}}(c/\eta^C_i, d/\eta^C_i) \\ &\Leftrightarrow \quad (e,f) \in \theta^C_{\mathcal{Q}}(c,d) \vee^{\mathcal{Q}} \eta^C_i. \end{aligned}$$

■

Remark 8. As in the variety case, a finitely generated relative-congruence quasivariety does not need to have DRPC. Let L be a finite non-distributive lattice and put $\mathcal{V} = \mathbf{HSP}(L)$, where $\mathbf{H}$ denotes the homomorphic image class operator. Jónsson's theorem [9, Corollary 3.4.] and congruence-distributivity of $\mathcal{V}$ yields $\mathcal{V}_{SI} \subseteq \mathbf{HS}(L)$, hence $\mathcal{V}$ is finitely generated as a quasivariety. But R. McKenzie proved [13, Theorem 23] that the only nontrivial variety of lattices with DPC is the variety of distributive lattices.

Proof of Pigozzi's theorem. Combine Theorem 2 and Proposition 7. ■

6. Examples

This section contains two examples, each providing a negative answer to questions emerging naturally in the context of our work.

Example 9. There is a finitely generated quasivariety $\mathcal{Q}$ without DRPSC such that the variety $\mathbf{H}(\mathcal{Q})$ generated by $\mathcal{Q}$ has DPSC. Let L be a finite lattice with $\mathbf{SP}(L)$ not finitely based. For examples of such lattices see [3] or [8, Sec. 6.2]. By Theorem 2, $\mathbf{SP}(L)$ does not have DRPSC. Still $\mathbf{HSP}(L)$ is congruence-distributive. Thus, by Theorem 2 in [2], it has DPSC.

Example 10. There is a finitely generated quasivariety $\mathcal{Q}$ with DRPSC such that the variety $\mathbf{H}(\mathcal{Q})$ generated by $\mathcal{Q}$ does not have DPSC. Let $A = S_3 \cup \{0\}$, where $(S_3, \cdot, 1)$ is the group of permutations of a three elements set and

$0 \notin S_3$. We extend the multiplication by $a0 = 0a = a$. We define two new operations

$$a + b = \begin{cases} 0 & \text{if } a = b = 0, \\ 1 & \text{otherwise;} \end{cases} \qquad \mathrm{eq}(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A = (A, \cdot, 1, 0, +, \mathrm{eq})$ is an equality-test algebra. It was shown in [19] that such algebras generate relatively congruence-distributive quasivarieties (see as well [15]). Thus $\mathbf{SP}(A)$ is relatively congruence-distributive quasivariety and by Proposition 7, it has DRPSC. Let α be the congruence of A with one nontrivial class $\{0, 1\}$. Then A/α is term equivalent to the group S_3 and by Theorem 3 in [2], $\mathbf{HSP}(A/\alpha)$ does not have DPSC. Thus $\mathbf{HSP}(A)$ does not have DPSC.

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