A CHARACTERIZATION OF VARIETIES EQUIVALENT TO VARIETIES OF AFFINE MODULES

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Abstract. In this paper we describe coproducts in varieties of affine modules. We prove that coproducts with described structure characterize, up to equivalence, varieties of affine modules.

1. Introduction

Any variety of algebras can be considered as a category with algebras as objects and homomorphisms as morphisms. Such category has coproducts. And it is interesting to know their detailed structure. This knowledge is relevant for a description of free algebras in any variety $\underline{\underline{V}}$ of algebras. It is well-known that the coproduct of free V-algebras X_iV over sets X_i, for $i \in I$, is a free V-algebra over the disjoint union of the sets X_i. Thus knowledge of "small" free algebras, for instance over one or two free generators, and a good structural description of coproducts, can provide a good description of any free V-algebra. This type of consideration was undertaken by Bela Csákány in [1], who has shown that under certain general conditions, the coproduct of any two algebras in a given variety $\underline{\underline{V}}$ coincides with their product iff $\underline{\underline{V}}$ is equivalent to a variety of semimodules. In this note we are interested in varieties in which the coproduct of any two algebras A and B is isomorphic to $A \times B \times 2V$, where $2V$ is a free V-algebra over two free generators. Such types of coproducts characterize varieties of affine modules. We show that a variety $\underline{\underline{V}}$ has coproducts of this type iff it is equivalent to a variety of affine modules. Varieties equivalent to varieties of affine modules were characterized by Bela Csákány [2] as idempotent, regular and hamiltonian varieties. Note that the existence of certain types of coproducts (so-called free products) of some families of algebras in a given variety was shown firstly by R. Sikorski [6]. Note also that in fact any variety of algebras considered as a category is also cocomplete.

At first we recall some necessary concepts and notation. Let A, B, C denote algebras in a variety $\underline{\underline{V}}$, or briefly V-algebras. The *coproduct* in $\underline{\underline{V}}$ of the V-algebras A and B is a V-algebra C together with homomorphisms $\rho_A : A \to C$ and $\rho_B : B \to C$ such that for any two homomorphisms $f_A : A \to D$ and $f_B : B \to D$ into a V-algebra D, there is exactly one homomorphism $f : C \to D$ such that $f \circ \rho_A = \rho_A f = f_A$ and $f \circ \rho_B = \rho_B f = f_B$. The coproduct C is denoted by $A \sqcup^V B$. Homomorphisms ρ_A and ρ_B are called *insertions*.

An algebra (A, Ω) is *idempotent*, if for each $\omega \in \Omega$, it satisfies the identity

$$x \ldots x\omega = x.$$

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A Mal'cev algebra A, i.e. an algebra admitting a Mal'cev term operation P, is *central* if the diagonal $D = \{(a, a) \mid a \in A\}$ is a class of a certain congruence of A^2. Equivalently, A is central if it satisfies the identities

$$x_1 y_1 z_1 P \ldots x_n y_n z_n P \; \omega = x_1 \ldots x_n \omega \, y_1 \ldots y_n \omega \, z_1 \ldots z_n \omega \; P \tag{C}$$

for each $\omega \in \Omega$. A variety $\underline{\underline{V}}$ is idempotent (central) if each V-algebra is idempotent (central).

Two algebras (A, Ω) and (A, Ψ), with the same universe, are *equivalent* if they have the same term operations. Two varieties $\underline{\underline{V}}$ and $\underline{\underline{W}}$, possibly of different types, are *equivalent* if they have the same derived operations. (Equivalently, $\underline{\underline{V}}$ and $\underline{\underline{W}}$ are equivalent if the free algebras $\mathbf{N}V$ and $\mathbf{N}W$ over countably infinite set of generators, are isomorphic to equivalent algebras.) Note that the notion of equivalence used here is stronger then categorical equivalence. See [5] or [4] for details. An algebra (A, Ψ) is a reduct of (A, Ω), if Ψ is a subset of the set of term operations of (A, Ω).

By an affine module over a ring[a)] R we understand the reduct $(A, \underline{R}, P)$ of a module $(A, +, R)$, where P is the Mal'cev operation $xyzP = x - y + z$ and $\underline{R}$ is the set of operations $xy\underline{r} = x(1 - r) + yr$. It is known that the operations $\underline{R}$ and P generate all idempotent term operations of each module $(A, +, R)$. The variety of modules over R is denoted by $\underline{\underline{Mod_R}}$. Also the class of affine modules over R forms a variety. It is denoted by $\underline{\underline{Aff_R}}$. It can be characterized similarly as in the case when the ring R is commutative (see [5, Section 6.3]).

Theorem 1. *The class $\underline{\underline{Aff_R}}$ is a central idempotent and Mal'cev variety. The free $\underline{\underline{Aff_R}}$-algebra on two generators (0 and 1) has the structure of the ring R. The variety $\underline{\underline{Aff_R}}$ is defined by the central, idempotent and Mal'cev identities and for all $p, q, r \in R$, the following additional ones:*

$$xy\underline{0} = x = yx\underline{1},$$
$$xy\underline{p}\, xy\underline{q}\, \underline{r} = xy\, \underline{pqr},$$
$$xy\underline{p}\, xy\underline{q}\, xy\underline{r}\, P = xy\, \underline{pqrP}.$$

Proof. It is analogous to the proof presented in [5, Section 6.3] in the case the ring R is commutative. One just has to replace entropicity for centrality. □

The next Theorem will be used later. It is also an easy generalization of corresponding theorems in [5, Section 6.3].

Theorem 2. *A variety $\underline{\underline{V}}$ is equivalent to the variety $\underline{\underline{Aff_R}}$ of affine modules over a ring R iff it is a central idempotent and Mal'cev variety.*

2. Coproducts in central idempotent and Mal'cev varieties

In this section we describe coproducts in central idempotent and Mal'cev varieties.

For a variety $\underline{\underline{V}}$, let $2V$ denote the free V-algebra over a two element set of free generators denoted by 0 and 1.

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a) In this paper the word ring means (not necessarily commutative) ring with identity.

Proposition 3. *Let $\underline{\underline{V}}$ be an idempotent central and Mal'cev variety. Then for any two V-algebras A and B, the algebra $C = A \times B \times 2V$ with maps $\rho_A : A \to C$; $a \mapsto (a, d, 0)$ and $\rho_B : B \to C$; $b \mapsto (c, b, 1)$, where c is any fixed element in A and d is any fixed element in B, is a coproduct of A and B in $\underline{\underline{V}}$.*

Proof. Since $\underline{\underline{V}}$ is idempotent, the maps ρ_A and ρ_B are homomorphisms. We will show that the set $X = A\rho_A \cup B\rho_B$ generates C. Let D be the subalgebra of C generated by X. First note that, by idempotency, $\{c\}\times\{d\}\times 2V \subseteq D$. Furthermore, by Mal'cev identities, $(a, d, u) = (a, d, 0)(c, d, 0)(c, d, u)\ P$. Hence $A\times\{d\}\times 2V \subseteq D$. Similar argument shows that $\{c\} \times B \times 2V \subseteq D$. And finally, since every element of C can be written as $(a, b, u) = (a, d, u)(c, d, u)(c, b, u)\ P$, it follows that $C \subseteq D$. Hence C and D coincide.

It remains to show that for any homomorphisms $f : A \to E$ and $g : B \to E$ into a V-algebra E, there exists a homomorphism $h : C \to E$, such that $\rho_A\ h = f$ and $\rho_B\ h = g$.

First note that every element of C can be written as

$$\begin{aligned}(a, b, u) &= (a_1, d, 0)\dots(a_n, d, 0)(c, b_1, 1)\dots(c, b_m, 1)\ w\\ &= (a_1 \dots a_n c \dots c\ w, d \dots d b_1 \dots b_m\ w, 0 \dots 0 1 \dots 1 w),\end{aligned}$$

where w is an $(n + m)$-ary term operation.

Then define a map h to be the following assignment

$$\begin{gathered}(a_1 \dots a_n c \dots c\ w, d \dots d b_1 \dots b_m\ w, 0 \dots 0 1 \dots 1 w)\\ \downarrow\\ a_1 f \dots a_n f b_1 g \dots b_m g\ w\end{gathered}$$

We will show that h is well defined. Assume therefore that

$$\begin{gathered}a_1 \dots a_n c \dots c\ w = a'_1 \dots a'_k c \dots c\ v,\\ d \dots d b_1 \dots b_m\ w = d \dots d b'_1 \dots b'_l\ v,\\ 0 \dots 0 1 \dots 1 w = 0 \dots 0 1 \dots 1 v.\end{gathered}$$

The third equality means that the identity $x \dots x y \dots y\ w = x \dots x y \dots y\ v$, where x occurs exactly n times on the left, k times on the right hand side and y occurs m times on the left, l times on the right hand side, holds in $\underline{\underline{V}}$. The equalities above imply the following:

$$\begin{aligned}&a_1 f \dots a_n f b_1 g \dots b_m g\ w\\ &\quad = (a_1 f\ dg\ dg\ P) \dots (a_n f\ dg\ dg\ P)(cf\ cf\ b_1 g\ P) \dots (cf\ cf\ b_m g\ P)\ w\\ &\quad = (a_1 \dots a_n c \dots c\, w\, f)(dg \dots dg\, cf \dots cf\ w)(d \dots d b_1 \dots b_m\, w\, g)\ P\\ &\quad = (a'_1 \dots a'_k c \dots c\, v\, f)(dg \dots dg\, cf \dots cf\ v)(d \dots d b'_1 \dots b'_l\, v\, g)\ P\\ &\qquad\qquad = c_1 f \dots c_k f d_1 g \dots d_l g\ v\ ,\end{aligned}$$

and this is what we wanted to show.

To finish the proof one should show that the map h is a homomorphism. This is quite standard and is left to the reader. □

Remark 4. There are at least two other possible proofs of Proposition 3. The first one is based on the fact that each central idempotent Mal'cev algebra is regular and hamiltonian. This approach was used in [8]. The second works nicely for varieties

$\underline{\underline{Aff_R}}$ and is based on the Sikorski's construction of coproducts, described in [6]. However, we believe that the proof presented here is the easiest and the most direct one.

3. A characterization of $\underline{\underline{Aff_R}}$

Here is the main result of the paper.

Theorem 5. *For any variety $\underline{\underline{V}}$ the following conditions are equivalent:*

(1) *$\underline{\underline{V}}$ is equivalent to a variety of affine modules;*
(2) *$\underline{\underline{V}}$ is a central idempotent and Mal'cev variety;*
(3) *$\underline{\underline{V}}$ has coproducts of the form described in Proposition 3;*
(4) *For each natural number n the algebra $(2V)^n$ is free over exactly $n+1$ free generators: $(0,\dots,0)$, $(1,0,\dots,0)$, $\dots$, $(0,\dots,0,1)$.*

Proof. Theorem 2 shows the equivalence ($1 \Leftrightarrow 2$).

Proposition 3 gives the implication ($2 \Rightarrow 3$).

Now we will focus our attention on the implication ($3 \Rightarrow 4$). The proof is by induction on n. First we show that the variety $\underline{\underline{V}}$ is idempotent i.e. the algebra $1V$ has exactly one element and thus is isomorphic to $(2V)^0$. If $\pi : A \times B \times 2V \to 2V$ is the third projection, then $A\rho_A\pi = \{0\}$ is a one-element subalgebra of $2V$. And hence $1V = \{0\}$. Then, by remarks in the introduction, $(n+1)V \cong 1V \sqcup^V nV \cong 1V \times nV \times 2V \cong 1V \times (2V)^{(n-1)} \times 2V \cong (2V)^n$. The elements $(0,\dots,0)$, $(1,0,\dots,0),\dots,(0,\dots,1)$ are the free generators of $(2V)^n$.

Now we prove the implication ($4 \Rightarrow 2$). Since $1V = \{0\}$, the variety $\underline{\underline{V}}$ is idempotent. Consider the free algebra $(2V)^2$ with the free generators $x = (1,0)$, $y = (0,0)$, $z = (0,1)$. Let P be a term operation, such that $xyzP = (1,1)$. Define the map $f : \{x,y,z\} \to (2V)^2$; $x, y \mapsto y$; $z \mapsto z$. Since $(2V)^2$ is free, the map f extends to the homomorphism $\overline{f} : (2V)^2 \to (2V)^2$; $(a,b) \mapsto (0,b)$. It follows that

$$xxzP = xfyfzf\ P = xyzP\ \overline{f} = (1,1)\overline{f} = z,$$

and similarly $xzzP = x$. Thus P is a Mal'cev term and $\underline{\underline{V}}$ is a Mal'cev variety. Now consider the free algebra $(2V)^3$ with the free generators $x = (1,0,0)$, $y = (0,0,0)$, $z = (0,1,0)$ and $t = (0,0,1)$. Then

$$\begin{aligned} xyzP\,t\,y\,P &= (1,0,0)(0,0,0)(0,1,0)P\,(0,0,1)\,(0,0,0)\,P \\ &= (1,1,0)(0,0,1)(0,0,0)P = (1,1,010P) \\ &= (1,0,0)(0,0,1)(0,1,0)P = xtzP. \end{aligned}$$

Next take the free algebra $nV \times nV$ with the free generators $x_i = (i,0)$, $y = (0,0)$, $z_j = (0,j)$, where $i,j = 1,\dots n$. Then for any basic operation ω of $nV \times nV$

$$\begin{aligned} x_1yz_1P\ \dots\ x_nyz_nP\ \omega &= (1,0)(0,0)(0,1)P\ \dots\ (n,0)(0,0)(0,n)P\ \omega \\ &= (1,1)\dots(n,n)\,\omega = (1\dots n\omega, 1\dots n\omega) \\ &= (1\dots n\omega, 0)(0,0)(0,1\dots n\omega)P \\ &= (1,0)\dots(n,0)\omega\ (0,0)\ (0,1)\dots(0,n)\omega\ P \\ &= x_1\dots x_n\omega\ y\ z_1\dots z_n\omega\ P. \end{aligned}$$

In this way we obtained the following identities:

(A) $$xyzP\,t\,y\,P = xtzP$$

(M) $$xyyP = x = yxxP$$

(QC) $$x_1yz_1P\ \ldots\ x_nyz_nP\ \omega = x_1\ldots x_n\omega\ y\ z_1\ldots z_n\omega\ P$$

Finally we use all the above identities to show centrality. If $x = x_1 \ldots x_n\omega$, $y = y_1 \ldots y_n\omega$ and $z = z_1 \ldots z_n\omega$, then

by (A) $xyzP = x_1\ldots x_n\omega\ t\,z_1\ldots z_n\omega\ P\ y\ t\ P$

by (QC) $= x_1tz_1P\ldots x_ntz_nP\,\omega\ y\ t\ P$

by (A) $= x_1y_1z_1P\,t\,y_1\,P\ldots x_ny_nz_nP\,t\,y_n\,P\,\omega\ y\ t\ P$

by (QC) $= x_1y_1z_1P\ldots x_ny_nz_nP\,\omega\ t\ y_1\ldots y_n\omega\ P\ y\ t\ P$

by (A) $= x_1y_1z_1P\ldots x_ny_nz_nP\,\omega\ y\ y\ P$

by (M) $= x_1y_1z_1P\ldots x_ny_nz_nP\,\omega.$

□

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