EMBEDDING SUMS OF CANCELLATIVE MODES INTO FUNCTORIAL SUMS

ANNA B. ROMANOWSKA[1], MICHAŁ STRONKOWSKI[2],
AND ANNA ZAMOJSKA-DZIENIO[3]

[1 2 3] Faculty of Mathematics and Information Science
Warsaw University of Technology
Plac Politechniki 1, 00-661 Warsaw, Poland
[2] Eduard Čech Center, Charles University, Prague, Czech Republic
[1] E-mail address: aroman@mini.pw.edu.pl
URL: http://www.mini.pw.edu.pl/˜aroman
[2] E-mail address: M.Stronkowski@mini.pw.edu.pl
[3] E-mail address: A.Zamojska@elka.pw.edu.pl

ABSTRACT. The paper discusses a representation of modes (idempotent and entropic algebras) as subalgebras of so-called functorial sums of cancellative algebras. We show that each mode that has a homomorphism onto an algebra satisfying a certain additional condition, with corresponding cancellative congruence classes, embeds into a functorial sum of cancellative algebras. We also discuss typical applications of this result.

1. Introduction

Algebras (A, Ω) considered in this paper have a *plural* type $\tau : \Omega \to \mathbb{Z}^+$, i.e. all operations of Ω are at least unary and at least one of them has arity bigger than one. If such an algebra (A, Ω) has a homomorphism h onto an idempotent algebra (I, Ω), then (A, Ω) is a disjoint union of its subalgebras $h^{-1}\{i\}$ for $i \in I$. If additionally the algebra (I, Ω) has a certain naturally defined quasi-order $\preceq$ (see Definition 2.1), and $i \mapsto h^{-1}\{i\}$ defines the object part of a functor from $(I, \preceq)$, considered as a small category, into the category of Ω-algebras, then the algebra (A, Ω) can be reconstructed from the fibres $h^{-1}\{i\}$ and the quotient (I, Ω) by means of a construction called a functorial sum [6, Ch. 4]. If the indexing algebra (I, Ω) is (equivalent to) a semilattice, the construction is known as a Płonka sum (and in the case of

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semigroups as a strong semilattice of semigroups). This construction is very useful for representing algebras in so-called regularized varieties. Recall that for an idempotent variety V, its regularization (or the regularized variety) is the variety $\widetilde{V}$, of the same type as V, defined by the regular identities true in V. Regular identities are characterized as those containing the same variables on both sides. Then each algebra in $\widetilde{V}$ is known to be a Płonka sum of subalgebras in V.

However, not all algebras can be represented as (non-trivial) Płonka or even as (non-trivial) functorial sums of subalgebras. The next class of interest concerns algebras that embed into functorial sums. (For a discussion of such embeddability, see [6, §4.5].) For example, each semigroup in the regularization $\widetilde{V}$ of an irregular variety V of semigroups is a subalgebra of a Płonka sum of V-semigroups [11, 12]. In the case of modes (idempotent and entropic algebras), it is known that a mode which decomposes into a sum of cancellative subalgebras, with a semilattice as its indexing algebra, embeds into a functorial sum of some cancellative algebras [4], [6, §7.4] with Errata [7], and the embedding is done in a simple, natural way. The proof of this result was based on the fact that each idempotent algebra (A, Ω) with a homomorphism h onto an algebra (I, Ω) that has a naturally defined quasi-order $\preceq$, may be reconstructed as so-called (coherent) Lallement sum of its subalgebras. Basic facts on Lallement and functorial sums are recalled in Section 3. The construction of a Lallement sum forms a generalization of a functorial sum, but it is not uniquely defined, and requires certain extensions of summands to define the operations on their union. It is not so elegant as the functorial sums, but may still be very useful for investigating the structure of algebras [4, 6, 8, 9, 10].

In this paper, we show that each mode has a natural quasi-order $\preceq$ (Theorem 2.2). This fact implies that each entropic algebra with a homomorphism onto an idempotent algebra is a Lallement sum of its fibres (Theorem 3.1). We use this result to prove that a Lallement sum of modes satisfying certain special cancellation laws, over a mode satisfying a certain additional general condition, embeds into a functorial sum of the summands (Theorem 4.2). This generalizes an earlier result concerning Lallement sums of cancellative modes over semilattices, and corrects a mistake in the formulation of Theorem 7.4.3 in [6]. (See Errata [7].) We investigate such sums more closely. We also discuss three typical situations when our results apply (Section 4). In the final section, we observe that our construction provides a representation for all modes in a quasivariety that is the Mal'cev product of a subquasivariety satisfying certain cancellativity laws and a regularized subvariety.

For more information about modes and their representations, we refer readers to the monographs [3, 6], and the papers provided in the references. We usually follow the notation and terminology of the two monographs.

2. Algebraic quasi-order of a mode

For a fixed type $\tau : \Omega \to \mathbb{Z}^+$ of plural algebras, let $X\Omega$ be the absolutely free τ-algebra over a countably infinite set X. The *translations* of a τ-algebra (A, Ω) are just unary polynomial operations of (A, Ω). More precisely, the i-translation of (A, Ω) determined by a word

$$x_1 \dots x_{i-1} x x_{i+1} \dots x_n w \in X\Omega$$

and an element $\mathbf{a} := (a_1, \dots, a_n) \in A^n$ is the mapping

$$w^i_{\mathbf{a}} : A \to A;\ b \mapsto a_1 \dots a_{i-1} b a_{i+1} \dots a_n w.$$

If there is no danger of confusion, or the place i is not essential, we will denote such translations simply by $w_{\mathbf{a}}$ and write $b\mathbf{a}w$ or $\mathbf{a}bw$.

For a given τ-algebra (A, Ω), define a binary relation $\preceq$ on A by: $a \preceq b$ if and only if $b = a\mathbf{c}w$ for some translation $w_{\mathbf{c}}$ of (A, Ω). One easily checks that this relation is a quasi-order.

Definition 2.1. The relation $\preceq$ is called the *algebraic quasi-order* of (A, Ω) [6, Section 4.1], [8]. If additionally the algebra (A, Ω) satisfies the condition

$$\text{if } a_i \preceq b_i, \text{ then } a_1 \dots a_n \omega \preceq b_1 \dots b_n \omega$$

for each (n-ary) $\omega \in \Omega$, and $a_1, \dots, a_n, b_1, \dots, b_n \in A$, then we say that the algebra is *naturally quasi-ordered*. If $\preceq = A \times A$, then the algebraic quasi-order is called *full*.

Note that the full quasi-order is natural.

Proposition 2.2. [6, Prop. 4.1.7] *Let (A, Ω) be an idempotent algebra with algebraic quasi-order $\preceq$. Then the following conditions are equivalent.*

(a) *(A, Ω) is naturally quasi-ordered;*

(b) *For each (n-ary) $\omega \in \Omega$, $a_1, \dots, a_n, a \in A$ and $i = 1, \dots, n$,*

$$\text{if } a_i \preceq a, \text{ then } a_1 \dots a_n \omega \preceq a;$$

(c) *The relation α defined on the set A by*

$$(a, b) \in \alpha \text{ if and only if } a \preceq b \text{ and } b \preceq a$$

is a congruence of (A, Ω), and the quotient (A^α, Ω) is an Ω-semilattice.

Recall that an Ω-semilattice is an τ-algebra equivalent to a semilattice. Note that in (c), the quasi-order $\preceq$ is full on each α-class.

Theorem 2.3. *Each τ-mode (A,Ω) is naturally quasi-ordered.*

Proof. Let $\omega \in \Omega$ be an n-ary operation. Assume that $a_1,\dots,a_n,a \in A$ and $a_i \preceq a$ for each $i = 1,\dots,n$. This means that there are words $xx_1\dots x_{k_i}t_i \in X\Omega$ and elements $\mathbf{b^i} = (b^i_1,\dots,b^i_{k_i}) \in A^{k_i}$ such that

$$a_i\mathbf{b^i}t_i = a$$

for each $i = 1,\dots,n$. We will show that

$$a_1\dots a_n\omega \preceq a_1\dots a_{n-1}a\omega \preceq \ \dots \ \preceq a_1a\dots a\omega \preceq a\dots a\omega = a. \tag{2.1}$$

First note that by the idempotency and entropicity

$$\begin{aligned} a = a\dots a\omega &= (a_1b^1_1\dots b^1_{k_1}t_1)a\dots a\omega \\ &= (a_1b^1_1\dots b^1_{k_1}t_1)(a\dots at_1)\dots(a\dots at_1)\omega \\ &= (a_1a\dots a\omega)(b^1_1a\dots a\omega)\dots(b^1_{k_1}a\dots a\omega)t_1, \end{aligned}$$

whence $a_1a\dots a\omega \preceq a$. Similarly for each $1 < m \leq n$,

$$\begin{aligned} &a_1\dots a_{m-1}a\dots a\omega = (a_1\dots a_1t_m)\dots \\ &\dots(a_{m-1}\dots a_{m-1}t_m)(a_mb^m_1\dots b^m_{k_m}t_m)(a\dots at_m)\dots(a\dots at_m) \\ &= (a_1\dots a_ma\dots a\omega)(a_1\dots a_{m-1}b^m_1a\dots a\omega)\dots \\ &\dots(a_1\dots a_{m-1}b^m_{k_m}a\dots a\omega)t_m, \end{aligned}$$

which shows that $(a_1\dots a_ma\dots a\omega) \preceq (a_1\dots a_{m-1}a\dots a\omega)$, and consequently proves (2.1). By transitivity we obtain

$$a_1\dots a_n\omega \preceq a.$$

By Proposition 2.2, the quasi-order $\preceq$ is natural. □

Proposition 2.4. *The algebraic quasi-order $\preceq$ of a mode (A,Ω) is full if and only if there is no homomorphism from (A,Ω) onto the two-element Ω-semilattice* **2**.

Proof. By Theorem 2.3, the algebraic quasi-order $\preceq$ of (A,Ω) is natural. If (A,Ω) has no homomorphism onto **2**, then its semilattice replica is trivial. Then by Proposition 2.2, the quasi-order $\preceq$ must be full.

Now let **2** be a join-semilattice defined on the two-element set $0 < 1$. Assume that $\preceq$ is full, but that there is an Ω-homomorphism $h : (A,\Omega) \to \mathbf{2}$ onto **2**. Let $a, x \in A$ be elements such that $ah = 1$, $xh = 0$ and $a \preceq x$. Then $x = a\mathbf{b}w$ for some translation $w_\mathbf{b}$ of (A,Ω). Hence $xh = ah\,\mathbf{b}h\,w = 1$, a contradiction. □

Modes with no homomorphisms onto **2**, are called *algebraically open*. See [6, Prop. 7.5.2] for other characterizations of algebraically open modes.

Recall that the quotient (A^θ, Ω) of an algebra (A, Ω) by a congruence θ is the Ω-*semilattice replica* of (A, Ω) if θ is the smallest congruence of (A, Ω) such that the quotient (A^θ, Ω) is an Ω-semilattice.

Corollary 2.5. *The quotient (A^α, Ω) of a mode (A, Ω) is the Ω-semilattice replica of (A, Ω). Moreover the quasi-order $\preceq$ restricted to each α-class of (A, Ω) is full.*

Proof. Suppose on the contrary that there is a semilattice congruence θ on (A, Ω) smaller than α, i.e. $\theta \neq \alpha$ and $\theta < \alpha$. Then there are $a, b \in A$ such that $(a, b) \in \alpha$ and $(a, b) \notin \theta$. Without loss of generality assume that $a^\theta < b^\theta$. Let $h = \operatorname{nat} \theta$ be the natural homomorphism determined by θ. Since $b \preceq a$, it follows that there are τ-word $x_0 x_1 \dots x_n\, t$ and $c_1, \dots, c_n \in A$ such that $a = bc_1 \dots c_n\, t = b\mathbf{c}t$, whence $ah = bh + c_1 h + \cdots + c_n h$ with all $c_i h \leq ah$ and $bh \leq ah$. This however gives a contradiction, since by our assumption, $ah = a^\theta < b^\theta = bh$. Hence $\theta = \alpha$. □

Corollary 2.6. *Let (A, Ω) be a mode. The algebraic quasi-order $\preceq$ of (A, Ω) is either full, or else (A, Ω) decomposes as the union of subalgebras (a^α, Ω), each with the full quasi-order $\preceq$, over its Ω-semilattice replica (A^α, Ω).*

3. Lallement sums of entropic algebras

A general construction of algebras we are interested in is the construction of a *generalized coherent Lallement sum of algebras* or briefly just a *Lallement sum*, as introduced and investigated in [3, 4, 6, 8]. The general context of the definition is the following. We are given a naturally quasi-ordered indexing algebra (I, Ω) with algebraic quasi-order $\preceq$, and for each i in I, an algebra (A_i, Ω). The algebras (A_i, Ω) come together with certain extensions (E_i, Ω), and for $i \preceq j$ in $(I, \preceq)$, there are Ω-homomorphisms $\varphi_{i,j} : (A_i, \Omega) \to (E_j, \Omega)$ with the mappings $\varphi_{i,i} : a_i \mapsto a_i$, and satisfying the following conditions

(L1) For each (n-ary) ω in Ω and for $i_1, \dots, i_n$ in I with $i_1 \dots i_n\, \omega = i$,
$$(A_{i_1} \varphi_{i_1,i}) \dots (A_{i_n} \varphi_{i_n,i})\, \omega \subseteq A_i;$$
(L2) For each $i_1 \dots i_n\, \omega = i \preceq j$ in $(I, \preceq)$,
$$a_{i_1} \varphi_{i_1,i} \dots a_{i_n} \varphi_{i_n,i}\, \omega\, \varphi_{i,j} = a_{i_1} \varphi_{i_1,j} \dots a_{i_n} \varphi_{i_n,j}\, \omega,$$
where $a_{i_k} \in A_{i_k}$ for $k = 1, \dots, n$;

(L3) $E_i = \{a_j \varphi_{j,i} \mid j \preceq i\}$.

Then the Lallement sum $\pounds_{i \in I}(A_i, \Omega)$, or simply $\pounds_{i \in I} A_i$, of A_i over I is the disjoint union $A = \biguplus\ (A_i \mid i \in I)$ equipped with Ω-operations defined as follows:
$$\omega : A_{i_1} \times \cdots \times A_{i_n} \to A_i;\ (a_{i_1}, \dots, a_{i_n}) \longmapsto a_{i_1} \varphi_{i_1,i} \dots a_{i_n} \varphi_{i_n,i}\, \omega$$

for each n-ary ω in Ω and $i = i_1 \dots i_n\omega$. The A_i are subalgebras of the sum (A, Ω), called the *sum fibres*, and there is an Ω-homomorphism

$$\pi : (A, \Omega) \to (I, \Omega);\ a_i \longmapsto i,$$

called a *projection*.

As proved in [8] (see also [6, Th. 4.5.3]), an algebra (A, Ω) with a homomorphism onto an idempotent, naturally quasi-ordered algebra (I, Ω), with corresponding fibres (A_i, Ω) for $i \in I$, is a Lallement sum $\pounds_{i\in I} A_i$ of the fibres (A_i, Ω) over (I, Ω). The extensions (E_i, Ω) are built in a certain canonical way as the so-called *envelopes* of the fibres: Each *preserves* the fibre subalgebra, in the sense that the equality relation is the only congruence on (E_i, Ω) preserving (A_i, Ω). Note that each entropic algebra has an idempotent replica (the largest idempotent homomorphic image). The replica is a mode, and by Theorem 2.3, is naturally quasi-ordered. This immediately implies the following theorem.

Theorem 3.1. *Let (A, Ω) be an entropic algebra with a homomorphism onto an idempotent algebra (I, Ω), with corresponding fibres (A_i, Ω), for $i \in I$. Then (A, Ω) is a Lallement sum $\pounds_{i\in I} A_i$ of (A_i, Ω) over (I, Ω).*

In other words, each entropic algebra is a Lallement sum of subalgebras over each idempotent homomorphic image. In particular, it is a Lallement sum of subalgebras over its idempotent replica.

In the case of modes, one obtains the following corollary.

Corollary 3.2. *Let $h : (A, \Omega) \to (I, \Omega)$ be a surjective mode homomorphism. Then (A, Ω) is a Lallement sum of the corresponding fibres over (I, Ω).*

Recall that in the case where (I, Ω) is (equivalent to) a semilattice, the sum $\pounds_{i\in I} A_i$ is called a *semilattice sum*. If $A_i = E_i$, for each $i \in I$, and the assignment $(i \preceq j) \mapsto (\varphi_{i,j} : A_i \to A_j)$ is a functor from the (small) category (I) to the category (Ω) of τ-algebras, the corresponding sum is called a *functorial sum*, and is denoted by $\sum_{i\in I}(A_i, \Omega)$ or just $\sum_{i\in I} A_i$. If additionally, $\sum_{i\in I} A_i$ is a semilattice sum, then the sum is a *Płonka sum*. Recall also that if the indexing algebra (I, Ω) of a functorial sum has a full algebraic quasi-order, all fibres A_i are isomorphic, and the sum is isomorphic to the direct product $(A_i \times I, \Omega)$.

Corollary 3.3. *Each mode, with the algebraic quasi-order $\preceq$, is a semilattice sum of its subalgebras with full quasi-order $\preceq$, over its semilattice replica.*

4. Embedding Lallement sums into functorial sums

In this section we investigate the problem of embedding Lallement sums of modes satisfying certain cancellation laws into functorial sums of algebras satisfying the same laws. Let a mode (A, Ω) be a Lallement sum $\pounds_{i\in I} A_i$ of modes (A_i, Ω) over a mode (I, Ω), for a fixed (plural) type $\tau : \Omega \to \mathbb{Z}^+$. Modes of a plural type will be called *plural.* Let t be a τ-word $x_1 \dots x_n y\, t$ with variables $x_1, \dots, x_n, y$, where $n \geq 1$, and linear with respect to y. (In particular, this means that y appears precisely once in t.) Assume that each (A_i, Ω), for $i \in I$, satisfies the following cancellation law:

$$x_1 \dots x_n y\, t = x_1 \dots x_n z\, t \to y = z. \tag{4.1}$$

We will say that the corresponding derived operation t is *cancellative* with respect to y or y-*cancellative*, and that the algebras (A_i, Ω) are $t(y)$-*cancellative.* Following [6, §7.4], let

$$P_j := \dot{\bigcup} \;(A_i \mid i \preceq j).$$

Define a relation $\mu = \mu(j)$ on P_j by:

$$(b_i, c_k) \in \mu :\Leftrightarrow\; \forall \mathbf{a} \in A_j^n,\; \mathbf{a}b_i\, t = \mathbf{a}c_k\, t,$$

where $i, k \preceq j$, moreover $b_i \in A_i$ and $c_k \in A_k$.

Lemma 4.1. *If for all $i, j \in I$ with $i \preceq j$, one has $j \dots j i\, t = j$, then μ is the largest congruence on (P_j, Ω) preserving (A_j, Ω). Moreover the envelope $(E_j, \Omega) = (P_j^{\mu}, \Omega)$ of (A_j, Ω) satisfies the cancellation law* (4.1)*, i.e. it is also $t(y)$-cancellative.*

Proof. The proof is very similar to the proof of [6, Lemma 7.4.1] with a correction provided in the Errata [7]. First, it is obvious that μ is an equivalence relation. Now for $i = 1, \dots, m$, let $k_i, l_i \preceq j$ and $b_i \in A_{k_i}, c_i \in A_{l_i}$. Assume that $(b_i, c_i) \in \mu$, i.e. for each $\mathbf{a} \in A_j^n$ one has $\mathbf{a}b_i\, t = \mathbf{a}c_i\, t$. Then the idempotent and entropic laws imply the following for each (m-ary) $\omega \in \Omega$:

$$\begin{aligned} \mathbf{a}(b_1 \dots b_m \omega)\, t &= \mathbf{a}b_1 t \dots \mathbf{a}b_m t\, \omega \\ &= \mathbf{a}c_1 t \dots \mathbf{a}c_m t\, \omega = \mathbf{a}(c_1 \dots c_m \omega)\, t, \end{aligned}$$

whence μ is a congruence of (P_j, Ω). The cancellation law (4.1) implies that μ preserves (A_j, Ω). Indeed, if $b, c \in A_j$ and $(b, c) \in \mu$, then $\mathbf{a}b\, t = \mathbf{a}c\, t$ implies $b = c$.

If λ is another congruence of (P_j, Ω) preserving (A_j, Ω) and $(b, c) \in \lambda$ for $b \in A_i$ and $c \in A_k$ with $i, k \preceq j$, then $(\mathbf{a}b\, t, \mathbf{a}c\, t) \in \lambda$ for each $\mathbf{a} \in A_j^n$. Since, by assumption, both these elements are in A_j and λ preserves (A_j, Ω), it follows that $\mathbf{a}b\, t = \mathbf{a}c\, t$, and hence $(b, c) \in \mu$.

Finally, we show that (E_j, Ω) satisfies the cancellation law (4.1). Let a_i be in A_{j_i} for $i = 1, \dots n$ and let $j_i \preceq j$. For $k, l \preceq j$, let $b \in A_k$ and $c \in A_l$. Assume that $(\mathbf{a}b\,t, \mathbf{a}c\,t) \in \mu$. Hence for each $\mathbf{d} \in A_j^n$ one has

$$\mathbf{d}(\mathbf{a}b\,t)\,t = \mathbf{d}(\mathbf{a}c\,t)\,t.$$

Applying the idempotent and entropic laws to both sides, one obtains

$$(\mathbf{d}a_1\,t) \dots (\mathbf{d}a_n\,t)(\mathbf{d}b\,t)\,t = (\mathbf{d}a_1\,t) \dots (\mathbf{d}a_n\,t)(\mathbf{d}c\,t)\,t.$$

Since, by assumption, all the elements in brackets are in A_j and (A_j, Ω) satisfies the cancellation law (4.1), it follows that $\mathbf{d}b\,t = \mathbf{d}c\,t$, whence $(b, c) \in \mu$. Consequently (E_j, Ω) satisfies (4.1), too. □

Theorem 4.2. *Let (A, Ω) be a Lallement sum $£_{i \in I} A_i$ of $t(y)$-cancellative modes (A_i, Ω), over a mode (I, Ω). If for all $i, j \in I$ with $i \preceq j$, one has $j \dots ji\,t = j$, then (A, Ω) embeds into a functorial sum of $t(y)$-cancellative envelopes (E_i, Ω) of (A_i, Ω) over the same indexing algebra (I, Ω).*

The proof follows by Lemma 4.1, in a way very similar to the proof of [6, Th. 7.4.2], with a correction provided in the Errata [7]. So we will omit it here.

Theorem 4.2 remains true in the case when the algebras (A_i, Ω) satisfy more than one cancellation law of the type (4.1). Assume that t_s, for $s \in S$, are τ-words determining y-cancellative operations on each (A_i, Ω).

Corollary 4.3. *Let (A, Ω) be a Lallement sum $£_{i \in I} A_i$ of modes (A_i, Ω), which are $t_s(y)$-cancellative for all $s \in S$, over a mode (I, Ω). If for all $i, j \in I$ with $i \preceq j$, one has $j \dots ji\,t = j$, for some fixed $t = t_s$, then (A, Ω) embeds into a functorial sum of envelopes (E_i, Ω) of (A_i, Ω), which are also $t_s(y)$-cancellative for all $s \in S$, over the same indexing algebra (I, Ω).*

Proof. Assume that all (A_i, Ω) satisfy (4.1), and also the same quasi-identity with the word t replaced by a word $w = x_1 \dots x_m y\,t_s$ for some $s \in S$. The proof that all (E_j, Ω) are $w(y)$-cancellative goes like the last part of the proof of Lemma 4.1. With $i = 1, \dots, m$ and the remaining notation as there, assume that $(\mathbf{a}b\,w, \mathbf{a}c\,w) \in \mu$. Hence for each $\mathbf{d} \in A_j^n$ one has

$$\mathbf{d}(\mathbf{a}b\,w)\,t = \mathbf{d}(\mathbf{a}c\,w)\,t.$$

Applying the idempotent and entropic laws to both sides, one obtains

$$(\mathbf{d}a_1\,t) \dots (\mathbf{d}a_m\,t)(\mathbf{d}b\,t)\,w = (\mathbf{d}a_1\,t) \dots (\mathbf{d}a_m\,t)(\mathbf{d}c\,t)\,w.$$

Since by assumption all the elements in brackets are in A_j and (A_j, Ω) satisfies the cancellation law (4.1) with w instead of t, it follows that $\mathbf{d}b\,t = \mathbf{d}c\,t$, whence $(b, c) \in \mu$. □

We consider three typical situations where the assumptions of Lemma 4.1 are satisfied, and hence Theorem 4.2 holds. The first concerns the case where (I, Ω) is (equivalent) to a semilattice, i.e. the corresponding Lallement sum is a semilattice sum. Then all derived (at least binary) operations of (I, Ω) are in fact semilattice operations. Denote the binary semilattice operation by $x + y$. Then each n-ary, with $n \geq 2$, semilattice operation is equal to $x_1 + \cdots + x_n$. For any word t as in Lemma 4.1, (I, Ω) satisfies $j \ldots j i\, t = j + \cdots + j + i = j + i$. Obviously $j + i = j$ precisely when $i \leq j$. (Recall that in this case $\leq$ and $\preceq$ coincide.) If all (A_i, Ω) are $t(y)$-cancellative, then the assumptions of Lemma 4.1 are satisfied and Theorem 4.2 holds.

Now recall that a mode (A, Ω) is *cancellative* if it satisfies the quasi-identity

$$(x_1 \ldots x_{i-1} y x_{i+1} \ldots x_n \omega = x_1 \ldots x_{i-1} z x_{i+1} \ldots x_n \omega) \to (y = z)$$

for each (n-ary) $\omega \in \Omega$ and each $i = 1, \ldots n$. In this case, one obtains the following.

Corollary 4.4. [6, Th. 7.4.2], [7] *If (A, Ω) is a semilattice Lallement sum $\pounds_{i \in I} A_i$ of cancellative modes (A_i, Ω) over an Ω-semilattice (I, Ω), then (A, Ω) embeds into a functorial sum of cancellative envelopes (E_i, Ω) over (I, Ω).*

The second case to be considered is the case where the indexing algebra (I, Ω) is in an irregular variety V of τ-modes. Consider an (at least binary) τ-word t as above. Let $t = w$ be an irregular identity true in V. Assume that y is a variable in t, but not in w. By substituting x for all variables different from y, one obtains an identity $x \circ y = x$ with two variables x and y that can be used as a unique irregular identity of a basis of V. (See e.g. [6, Ch. 4].) Clearly, for any $i, j \in I$, we have $j \circ i = j$, and the corresponding algebraic quasi-order of (I, Ω) is full. If all (A_i, Ω) satisfy the law (4.1), the assumptions of Theorem 4.2 are satisfied, and we obtain the following corollary.

Corollary 4.5. *Let (A, Ω) be a Lallement sum $\pounds_{i \in I} A_i$ of $t(y)$-cancellative modes (A_i, Ω). Let the indexing algebra (I, Ω) satisfy an irregular identity $t = w$, where y is a variable of t but not of w, and t is linear with respect to y. Then (A, Ω) embeds into a functorial sum of $t(y)$-cancellative envelopes (E_i, Ω) over the indexing algebra (I, Ω).*

Since in this case the indexing algebra (I, Ω) has a full algebraic quasi-order, all envelopes (E_i, Ω) are isomorphic, say to (E, Ω), and the functorial sum of (E_i, Ω) over (I, Ω) reduces to the direct product $(E, \Omega) \times (I, \Omega)$ [1], [6, Ch. 4]. This implies the following corollary.

Corollary 4.6. *If (A, Ω) is a Lallement sum $\pounds_{i\in I}A_i$ as in Corollary 4.5, then all the envelopes (E_i, Ω) are isomorphic, say to (E, Ω), and (A, Ω) embeds into the direct product $(E, \Omega)\times(I, \Omega)$ of the common envelope (E, Ω) and the indexing algebra (I, Ω).*

The third case generalizes the two previous ones. Now we assume that the algebra (I, Ω) belongs to the regularization $\widetilde{V}$ of the irregular variety V considered in the previous case. In such a case, (I, Ω) is a Płonka sum of V-algebras, the derived operation $x \circ y$ becomes a left normal band operation, and for $i \preceq j$, one has $j \circ i = j$. Again, if all the fibres of the sum satisfy (4.1), the assumptions of Theorem 4.2 are satisfied, and we obtain the following corollary.

Corollary 4.7. *Let (A, Ω) be a Lallement sum $\pounds_{i\in I}A_i$ of $t(y)$-cancellative modes (A_i, Ω) over a mode (I, Ω) in the regularization $\widetilde{V}$ of an irregular variety V as above. Then (A, Ω) embeds into a functorial sum $\sum_{i\in I} E_i$ of $t(y)$-cancellative envelopes (E_i, Ω) over (I, Ω).*

Let the algebra (I, Ω) in Corollary 4.7 be a Płonka sum of V-algebras (I_s, Ω) over a semilattice (S, Ω). By [10, Th. 3.2], the functorial sum $\sum_{i\in I} E_i$ may be expressed as

$$\sum_{i\in I} E_i = \sum_{s\in S}(\sum_{i\in I_s} E_i).$$

Moreover, since the algebraic quasi-order of each I_s is full, it follows that all the summands (E_i, Ω) of the subalgebra $B_s = \sum_{i\in I_s} E_i$ are isomorphic, say to (E_s, Ω), and $(B_s, \Omega) \cong (E_s, \Omega) \times (I_s, \Omega)$. Consequently

$$\sum_{i\in I} E_i = \sum_{s\in S} B_s = \sum_{s\in S}(E_s \times I_s).$$

Note that a τ-mode which is $t(y)$-cancellative for all τ-words $\mathbf{x}y\,t$ linear with respect to y is cancellative. Hence Corollaries 4.5, 4.6, and 4.7 also hold for cancellative τ-modes. In particular, we obtain the following corollary.

Corollary 4.8. *Let (A, Ω) be a Lallement sum $\pounds_{i\in I}A_i$ of cancellative modes (A_i, Ω) over a mode (I, Ω) in the regularization $\widetilde{V}$ of an irregular variety V as above. Then (A, Ω) embeds into a functorial sum $\sum_{i\in I} E_i$ of cancellative envelopes (E_i, Ω) over (I, Ω).*

We provide an example of a mode I that does not satisfy the condition of Lemma 4.1, as an indexing algebra, for every Ω-word $x_1 \dots x_n y\,t$ linear with respect to y. Let $I = (\mathbb{Z}_{10}, *)$, where $i * j = 2i - j$. Because the identity $x * (x * y) = y$ is valid in I, the mode I has full algebraic quasi-order. In particular, $3 \preceq 0$. But $0 \dots 03\,t \neq 0$ for every $\{*\}$-word $x_1 \dots x_n y\,t$ linear with respect to y.

Note that embeddability of Lallement sums of cancellative modes over a mode that does not satisfy the condition of Lemma 4.1, still remains as an open problem.

5. Quasivarieties of Lallement sums

By the results of the previous section, one may easily deduce that Lallement sums of modes of the kind considered there form certain special quasivarieties.

First note that if a mode (A, Ω) is cancellative, then many of its derived operations are also cancellative. Cancellativity for derived operations of (A, Ω) is defined as in the case of basic operations, i.e. it concerns linear derived operations. However, this definition may easily be extended to the case of derived operations determined by words of the form $x_1 \dots x_k y_1 \dots y_l\, t$, where t is linear with respect to each y_i. This assumption is essential. For example, consider the variety Q of quasigroups $(A, \cdot, /, \backslash)$, and recall that the basic operations of quasigroups are cancellative. Then Q satisfies the quasi-identity $(y/x)x = (z/x)x \to y = z$, but not the quasi-identity $(xy)/y = (xz)/z \to y = z$.

Lemma 5.1. *The quasivariety $Cl(\Omega)$ of cancellative plural τ-modes satisfies all cancellation laws* (4.1) *for all at least binary τ-words t, linear with respect to y.*

Proof. First note that $Cl(\Omega)$ satisfies all the $t(y)$-cancellation laws obtained from the cancellativity of the basic operations by identifying some variables different from y. (We may assume without loss of generality that each basic operation contains the variable y). Then assume that the proposition holds for τ-words of length smaller than n. Let $\mathbf{x}y\, t$ and $\mathbf{z}y\, w$ be such words. In particular, this means that the following quasi-identities hold:

$$\mathbf{x}y\, t = \mathbf{x}y'\, t \to y = y' \tag{5.1}$$

and

$$\mathbf{z}y\, w = \mathbf{z}y'\, w \to y = y'. \tag{5.2}$$

Assume that the word

$$\mathbf{x}(\mathbf{z}y\, w)\, t$$

has length n. We will show that the corresponding derived operation t is y-cancellative. First note that by the idempotent and entropic laws

$$\mathbf{x}(\mathbf{z}y\, w)\, t = (\mathbf{x}z_1\, t)\ \dots\ (\mathbf{x}z_j\, t)\, (\mathbf{x}y\, t)\, w. \tag{5.3}$$

Then (5.2) implies that the quasi-identity

$$(\mathbf{x}z_1\, t)\ \dots\ (\mathbf{x}z_j\, t)\, (\mathbf{x}y\, t)\, w = (\mathbf{x}z_1\, t)\ \dots\ (\mathbf{x}z_j\, t)\, (\mathbf{x}y'\, t)\, w \to \mathbf{x}y\, t = \mathbf{x}y'\, t$$

holds in $Cl(\Omega)$. By (5.1), transitivity and (5.3), the $t(y)$-cancellation law

$$\mathbf{x}(\mathbf{z}y\,w)\,t = \mathbf{x}(\mathbf{z}y'\,w)\,t \rightarrow y = y'$$

holds in $Cl(\Omega)$, as well. □

By Lemma 5.1, the quasivariety $Cl(\Omega)$ of τ-modes is also defined by all $t(y)$-cancellativities. Subsets of the set of all $t(y)$-cancellative laws define quasivarieties of τ-modes containing $Cl(\Omega)$. We will call them *cancellative quasivarieties*. Under inclusion, cancellative quasivarieties form an ordered set.

Recall that the *Mal'cev product* $\mathcal{K}_1 \circ \mathcal{K}_2$ of two classes $\mathcal{K}_1$ and $\mathcal{K}_2$ of τ-modes consists of τ-modes with quotients in $\mathcal{K}_2$ and corresponding congruence classes in $\mathcal{K}_1$. Mal'cev products of quasivarieties of modes are again quasivarieties. More generally, the Mal'cev product of subquasivarieties of a variety of modes is again a subquasivariety. (See e.g. [2] and [6, §3.7] for more general results.) For a fixed τ-word $t = \mathbf{x}y\,t$, linear with respect to y, let $C_{t(y)}(\Omega)$ be the quasivariety of $t(y)$-cancellative τ-modes. Let $I_{t(y)}$ be the variety of τ-modes defined by a set of regular identities and the identity $x \ldots xy\,t = x$. Let $\widetilde{I_{t(y)}}$ be its regularization. Finally, let Sl be the variety of Ω-semilattices. Corollaries 4.4, 4.5, 4.7, and 4.8 provide representations of modes in the quasivarieties $Cl(\Omega) \circ Sl$, $C_{t(y)}(\Omega) \circ I_{t(y)}$, $C_{t(y)}(\Omega) \circ \widetilde{I_{t(y)}}$, and $Cl(\Omega) \circ \widetilde{I_{t(y)}}$ as Lallement sums of cancellative or $t(y)$-cancellative τ-modes over the corresponding quotients. Note that Corollaries 4.5 and 4.7 may easily be extended by replacing $C_{t(y)}(\Omega)$ with any cancellative quasivariety of τ-modes satisfying $t(y)$-cancellativity, using Corollary 4.3. We summarize the above remarks as the following corollary.

Corollary 5.2. *Let $t = \mathbf{x}y\,t$ be a τ-word, linear with respect to y. Let Q be a cancellative quasivariety of plural τ-modes satisfying $t(y)$-cancellativity, and let $\widetilde{V}$ be the regularization of a variety of τ-modes satisfying the identity $x \ldots xy\,t = x$. Then each algebra in the Mal'cev product $Q \circ \widetilde{V}$ of modes is a subalgebra of a functorial sum of Q-modes over a $\widetilde{V}$-mode.*

Cancellative quasivarieties of plural τ-modes deserve further detailed investigations. In particular, it is not clear at the moment if they form a sublattice of the lattice of quasivarieties of τ-modes.

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