LECTURE 2.

- Propositional functions
- Quantifiers
- Generalized set operations

Definition. A *propositional function* (*a predicate*) is an expression which contains one or more *free variables*, i.e. such variables that assigning specific values to those variables turns the expression into a statement. In other words, a propositional function is a function from some set X into a set of propositions.

Examples.

1. $x > 7$ is a propositional function defined on $\mathbb{R}$ (or another set of numbers. Replacing x with a number turns this expression into a (true or false) statement.
2. $x > y$ is a propositional function defined on $\mathbb{R} \times \mathbb{R}$.
3. $x^n > 0$ is a propositional function defined on $\mathbb{R} \times \mathbb{Z}$.
4. *"For every x, $x > y$"* is a propositional function defined on $\mathbb{R}$. y is a free variable here; x is a variable, but it has no effect on the logical value (if any) of the expression. x is said to be *bound* by the phrase "For every".

We introduce two new symbols called *quantifiers*:

$\forall$, meaning *for all* or *for every*. It is called the *general (or universal) quantifier*.

$\exists$, meaning *there exists*. It is called *the existential quantifier.*

Given a propositional function $\varphi(x)$ defined on a set X we may write

- $(\forall x \in \mathrm{X})\varphi(x)$ (for every element x in X it is true that $\varphi(x)$)
- $(\exists x \in \mathrm{X})\varphi(x)$ (there is an element x in X for which $\varphi(x)$)

For example:

$(\forall x \in \mathbb{R})(x > 7)$ is a statement, a false one

$(\exists x \in \mathbb{R})(x > 7)$ is a true statement

$x > y$ is a propositional function defined on $\mathbb{R} \times \mathbb{R}$)*. It has two free variables. We can turn it into a statement in various ways:

1. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x > y)$ – false
2. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x > y)$ – true
3. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x > y)$ – true
4. $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x > y)$ – false

There is an important lesson to be learned here:

Changing the order of quantifiers may affect the logical value of our statement!

The same applies to

5. $(\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x > y)$ – true
6. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x > y)$ – false

Incidentally, it should be "φ is a propositional function such that $\varphi(x,y)=x>y$"; $x>y$ is the value of φ for some x and y.

On the other hand, in $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\varphi(x, y)$ and in $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})\varphi(x, y)$the order of quantifiers makes no difference, and such expressions are usually shortened to $(\forall x, y \in \mathbb{R})\varphi(x, y)$ and $(\exists x, y \in \mathbb{R})\varphi(x, y)$.

In order to avoid ambiguities, the propositional function a quantifier applies to should be enclosed in parenthesis which define *the range* of the quantifier. For example writing

$$(\forall x \in \mathrm{X})\varphi(x) \wedge \psi(x)$$

might mean $[(\forall x \in \mathrm{X})\varphi(x)] \wedge \psi(x)$, which does not make much sense but is formally correct, or $(\forall x \in \mathrm{X})(\varphi(x) \wedge \psi(x))$, which is what we would expect.

In general, using one variable letter in two different meanings, while formally acceptable, is asking for trouble both in logic and in programming.

Remarks.

1. $\forall$ is a generalization of conjunction. If X={1,2,3,4,5}, instead of (1>0 $\wedge$ 2>0 $\wedge$ 3>0 $\wedge$ 4>0 $\wedge$ 5>0) we may say $(\forall x \in X)\ (x > 0)$.
2. In the same sense, $\exists$ is a generalization of disjunction:
(1>3 $\vee$ 2>3 $\vee$ 3>3 $\vee$ 4>3 $\vee$ 5>3) $\equiv$ $(\exists y \in X)(y > 3)$
3. The name of a variable bound by a quantifier does not matter:
$(\exists y \in X)\varphi(y) \equiv (\exists z \in X)\varphi(z)$
$(\forall y \in X)\varphi(y) \equiv \left(\forall \text{و} \in X\right)\varphi\left(\text{و}\right)$.
4. We can re-define propositions: *a proposition is a declarative sentence containing no free variables*.

Quantifiers and sets

Recall that when we discuss sets, we assume that all considered sets are subsets of some universal set X.

We can specify a particular subset A of X by listing all its elements in curly brackets, but we can also say that A is the set of those and only those elements of X that satisfy some condition.

It looks like this:

$$\mathrm{A} = \{x \in X : \varphi(x)\}$$

where φ is a propositional function defined on X.

Examples.

A = $\{n \in Z\colon \neg(2|n)\}$ where | denotes the divisibility relation (2 divides n). A is the set of all odd integers (we use colon here instead of | to avoid double meaning of |)

$\mathbb{R}^{+} = \{d \in \mathbb{R}|d > 0\}$, here $\varphi(x) = x>0$

$\emptyset = \{c \in \mathbb{R}|c^2 < 0\}$, here $\varphi(x) = x^2 < 0$

$C_{0,1} = \{(x, s) \in \mathbb{R}^2|x^2 + s^2 = 1\}$, here the condition is a propositional function of two variables, $\varphi(p, q) = (p^2 + q^2 = 1)$. $C_{0,1}$ is the unit circle

$D_{0,1} = \{(x, s) \in \mathbb{R}^2|x^2 + s^2 \leq 1\}$ here $\varphi(z, \text{و}) = (z^2 + \text{و}^2 = 1)$. $D_{0,1}$ is the unit disc.

Comprehension.

Define the set of primes in this way (use only mathematical and logical symbols, including quantifiers if need be).

We used this notation when we defined set operations:

- A∪B = $\{x \in X | x \in \mathrm{A} \vee x \in B\}$
- A∩B = $\{x \in X | x \in \mathrm{A} \wedge x \in B\}$
- A\B = $\{x \in X | x \in \mathrm{A} \wedge x \notin B\}$
- A' = $\{x \in X | x \notin A\}$.

Let A $= \{x \in X: \varphi(x)\}$ and let B $= \{x \in X: \psi(x)\}$. Then

- A∪B = $\{x \in X: \varphi(x) \vee \psi(x)\}$
- A∩B = $\{x \in X: \varphi(x) \wedge \psi(x)\}$
- A\B = $\{x \in X: \varphi(x) \wedge \neg\psi(x)\}$
- A' = $\{x \in X: \neg\varphi(x)\}$.

Quantifiers can be defined in terms of sets:

$(\forall x \in \mathrm{X})(\varphi(x)) \equiv \{q \in X : \varphi(q)\} = X.$

$(\exists x \in \mathrm{X})\big(\varphi(x)\big) \equiv \{t \in X : \varphi(t)\} \neq \emptyset.$

Since quantifiers generalize conjunction and disjunction some laws of propositional calculus should apply to quantifiers. "Some" because many, like commutativity and associativity of conjunction and disjunction, are meaningless. Some other are just as important in predicate calculus as they are in propositional calculus.

Theorem. (De Morgan's Law for quantifiers)
For every predicate φ defined on a set X

1. $\neg\big((\forall x \in X)\varphi(x)\big) \equiv (\exists t \in X)(\neg\varphi(t))$
2. $\neg\big((\exists y \in X)\varphi(y)\big) \equiv (\forall q \in X)(\neg\varphi(q))$

Proof.
Part 2. $\neg\big((\exists x \in X)\varphi(x)\big)$ means $A = \{x \in X : \varphi(x)\} = \emptyset$. Since A is a subset of X, A' = X. Now, A' = $\{x \in X | x \notin A\} = \{x \in \text{X}: \neg\varphi(x)\}$. Hence, X = $\{x \in \text{X}: \neg\varphi(x)\}$, which means that

$$(\forall x \in X)(\neg\varphi(x)).$$

Part 1. We negate both sides of 2., with $\varphi(x)$ replaced with $\neg\varphi(x)$
$\neg\neg\big((\exists y \in X)\neg\varphi(y)\big) \equiv \neg(\forall q \in X)\neg\Big(\neg\big(\varphi(q)\big)\Big)$. From the double negation law we obtain
$(\exists y \in X)\neg\varphi(y) \ \equiv \neg((\forall q \in X)\varphi(q))$. QED

Example.

Consider a function f with the following property:

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(|x| < \delta \Rightarrow |f(x)| < \varepsilon)$

Soon enough you will learn that this means "f approaches 0 as *x* approaches 0". How do we write "it is not true that f approaches 0 as *x* approaches 0"? Well, we negate the whole expression:

$\neg[(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(|x| < \delta \Rightarrow |f(x)| < \varepsilon)] \equiv$

$(\exists \varepsilon > 0)\neg[(\exists \delta > 0)(\forall x \in \mathbb{R})(|x| < \delta \Rightarrow |f(x)| < \varepsilon)] \equiv$

$(\exists \varepsilon > 0)(\forall \delta > 0)\neg[(\forall x \in \mathbb{R})(|x| < \delta \Rightarrow |f(x)| < \varepsilon)] \equiv$

$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\neg(|x| < \delta \Rightarrow |f(x)| < \varepsilon) \equiv$

(using $p \Rightarrow q \equiv \neg p \vee q$)

$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\neg[\neg(|x| < \delta) \vee |f(x)| < \varepsilon)] \equiv$

$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\ (|x| < \delta) \wedge |f(x)| \geq \varepsilon)]$

Theorem.

For every two formulas φ and ψ defined on some set X

1. $(\forall x)(\varphi(x) \wedge \psi(x)) \Leftrightarrow ((\forall x)\, \varphi(x) \wedge (\forall x)\, \psi(x))$
2. $(\exists x)(\varphi(x) \vee \psi(x)) \Leftrightarrow ((\exists x)\, \varphi(x) \vee (\exists x)\, \psi(x))$
3. $((\forall x)\, \varphi(x) \vee (\forall x)\, \psi(x)) \Rightarrow (\forall x)(\varphi(x) \vee \psi(x))$
4. $(\exists x)(\varphi(x) \wedge \psi(x)) \Rightarrow ((\exists x)\, \varphi(x) \wedge (\exists x)\, \psi(x))$

Proof.

1. ($\Rightarrow$, by contradiction) Suppose RHS is false. By de Morgan Law, there is p such that $\varphi(p)$ is false (then $\varphi(p) \wedge \psi(p)$ is false) or, there is q such that $\psi(q)$ is false, hence $\varphi(q) \wedge \psi(q)$ is false. In both cases the LHS is false.

($\Leftarrow$, by contradiction) . If there is *t* such that $(\varphi(t) \wedge \psi(t))$ is false, then $\varphi(t)$ is false OR $\psi(t)$ is false, so RHS is false.

Comprehension.

1. Prove the remaining parts.
2. Can we replace $\Rightarrow$ with $\Leftrightarrow$ in part 3 and/or part 4?

Laws of set algebra.

The set operations of union and intersection are very closely related to disjunction and conjunction, the set complement operation plays the part of negation and $\emptyset$ and X act in the same capacity as logical constants 0 and 1, respectively. It is only to be expected that laws of propositional calculus have their counterparts in set algebra. Set equality replaces logical equivalence.

Theorem.

For every three subsets A,B and C of X

1. commutativity law:
   $A \cap B = B \cap A, \;\; A \cup B = B \cup A$
2. associativity law:
   $(A \cap B) \cap C = A \cap (B \cap C), (A \cup B) \cup C = A \cup (B \cup C)$
3. distributivity law:
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. $A \cup A' = X$
5. $A \cap A' = \emptyset$
6. absorption law: $A \cap (B \cup A) = \text{A}, \;\; A \cup (B \cap A) = \text{A}$
7. double complement: $(A')' = A$

There are some differences, though. We do not use the set analogue of the conditional connective IF … THEN (implication). Implication is used in the definition of the inclusion relation $\subseteq$.

$$A \subseteq B \equiv (\forall x \in X)(x \in A \Rightarrow x \in B)$$

Definition.

The *symmetric difference* of sets A and B is defined as follows:

$$A \div B = (A \setminus B) \cup (B \setminus A)$$

Properties.

1. $A \div B = (\mathrm{A} \cup B) \setminus (A \cap B)$
2. $A \div B$ consists of those elements of X who belong to exactly one of the two sets A and B
3. $A \div B = B \div A$
4. $(A \div B) \div C = A \div (B \div C)$
5. $A \div \emptyset = \emptyset \div A = A$ ($\emptyset$ is the identity element for $\div$)
6. $A \div A = \emptyset$ (every set is its own inverse element)

The following slides were NOT presented on Oct 15 lecture

Generalized set operations

Suppose that with every element of some set I (called the *set of indices*) we associate a subset of our universal set. For example we can consider sets A_1, A_2, A_3 and our I = {1,2,3} but we can just as well put $A_t=[-t;t]$ and let t range over [0;∞). Such a construction is called an *indexed family of subsets* of X. We can generalize the idea of union (intersection) of two sets to any indexed family of sets.

Definition.

The *generalized union of the family* $\{A_i\}_{i\in I}$ is the set

$$\bigcup_{i\in I} A_i = \{x \in X | (\exists i \in I) x \in A_i\}$$

The *generalized intersection of the family* $\{A_i\}_{i\in I}$ is the set

$$\bigcap_{i\in I} A_i = \{x \in X | (\forall i \in I) x \in A_i\}$$

The expressions look complicated but what they really say is that

x belongs to $\bigcup_{i \in I} A_i$ if and only if it belongs to at least one set from the collection of sets $\{A_i\}_{i \in I}$

and

x belongs to $\bigcap_{i \in I} A_i$ if and only if it belongs to each set from the collection of sets $\{A_i\}_{i \in I}$.

For two-element families of sets this is clearly ordinary union and intersection: $\{A_i\}_{i \in \{1,2\}} = \{A_1, A_2\}$, $\bigcup_{i \in \{1,2\}} A_i = A_1 \cup A_2$ and $\bigcap_{i \in \{1,2\}} A_i = A_1 \cap A_2$.

Example.

Consider our example of $\{A_t\}_{t\in[0;\infty)}$ and $A_t = [-t;t]$. Obviously, the only number belonging to each of these closed intervals is 0, hence $\bigcap_{i\in[0;\infty)} A_i = \{0\}$.

On the other hand, every real number x belongs to some closed interval $[-t;t]$ (for example $x \in [-x;x]$), hence $\bigcup_{i\in[0;\infty)} A_i = \mathbb{R}$.