LECTURE 3.

- Generalized set operations
- Relations
- Properties of relations

Generalized set operations

Suppose that with every element of some set I (called the *set of indices*) we associate a subset of our universal set. For example we can consider some three sets A_1, A_2, A_3 using I = {1,2,3}, but we can just as well put $A_i=[-i;i]$ and I = $[0;\infty)$. Such a construction is called an *indexed family of subsets* of X and is denoted by $\{A_i\}_{i\in I}$. We generalize the concept of union (intersection) of two sets to any indexed family of sets.

Definition.

The *generalized union of the family* $\{A_i\}_{i\in I}$ is the set

$$\bigcup_{i\in I} A_i = \{x \in X | (\exists i \in I) x \in A_i\}$$

The *generalized intersection of the family* $\{A_i\}_{i\in I}$ is the set

$$\bigcap_{i\in I} A_i = \{x \in X | (\forall i \in I) x \in A_i\}$$

The expressions look complicated but what they really say is that

x belongs to $\bigcup_{i\in I} A_i$ if and only if it belongs to at least one set from the collection $\{A_i\}_{i\in I}$

and

x belongs to $\bigcap_{i\in I} A_i$ iff it belongs to each set from $\{A_i\}_{i\in I}$.

For two-element families of sets this reduces to ordinary union and intersection: $\{A_i\}_{i\in\{1,2\}} = \{A_1, A_2\}$, $\bigcup_{i\in\{1,2\}} A_i = A_1 \cup A_2$ and $\bigcap_{i\in\{1,2\}} A_i = A_1 \cap A_2$.

Example.

Consider our example of $\{A_t\}_{t\in[0;\infty)}$ and $A_t = [-t;t]$. Obviously, the only number belonging to each of these closed intervals is 0, hence $\bigcap_{i\in[0;\infty)} A_i = \{0\}$.

On the other hand, every real number x belongs to some closed interval $[-t;t]$ (for example $x \in [-x;x]$), hence $\bigcup_{i\in[0;\infty)} A_i = \mathbb{R}$.

Relations

When dealing with sets we often distinguish some elements as having a particular property. The "property" can be described analytically, like "having 3 as the reminder from division by 5" could be a property of integers. Some integers do and other don't have this property. Clearly the property defines a subset of X, namely the set consisting of all elements of X that have the property (or satisfy the condition). It works the other way around, given a subset A of X we can always construct a condition (property) φ such that A is the set of all elements of X which satisfy the φ (even if sometimes it is a silly-looking one like $\varphi(x) = (x \in A)$). There is a very close relationship between subsets of X and properties of elements of X.

The word "relation" is used in mathematics in the meaning of "a property of pairs of elements". Hence, the following definition:

Definition.
Given any two sets X and Y, a *relation* R between elements of X and elements of Y is a subset of X×Y.

X and Y may be equal or different.

If X=Y, we say that R is a *relation on X*.

If R=∅ we call R the *empty relation*. It is not particularly exciting, nobody in X is related to anybody in Y. On the other end of the spectrum we have the *full relation* R=X×Y, where everybody in X is related to everybody in Y – not very exciting either.

Traditionally, instead of writing $(x,y) \in R$ we write xRy, as in x<y (for numbers), $p|q$ (for integers) or $A \subseteq B$ for sets.

Suppose R and S are two relations between sets X and Y. Since relations are sets (subsets of $X \times \mathrm{Y}$), expressions like $R \cup S, R \cap S, \mathrm{R}'$ or $R \setminus S$ or even $R \subseteq S$ make perfect sense.

For example, $x(R \cup S)y \equiv xRy \vee xSy$ because $x(R \cup S)y$ means $(x, y) \in R \cup S$, which means $(x, y) \in R$ or $(x, y) \in S$, i.e. $xRy \vee xSy$.

EBE (Even Better Example):
Given two relations on $\mathbb{R}$, $\leq$ and $\geq$, what is $\leq \cap \geq$? Looks silly, doesn't it? The answer looks even more silly: $\leq \cap \geq = =$. To avoid silly-looking answers we often use extra symbols to distinguish between the equality relation on some set and plain saying that something is the same as something else. I could introduce LTE for "less than or equal to" relation, "GTE" for "greater than or equal to" and "EQ" for the equality relation. Then we say $LTE \cap \mathrm{GTE} = EQ$ which looks much better.

YAE (Yet Another Example)
We use X= $\mathbb{R}$. The expression $\leq=<\cup=$ makes perfect sense. Well, perhaps it does if we use the naming convention from the last slide: LTE=LT∪EQ. Thus literarily means "less than or equal is the same as less than or equal". What can be more true than $p \equiv p$?

We can also say things like $LT \subseteq LTE$ or $LT \cap GT = \emptyset$ and so on.

First, we will study relations on one set, i.e. we will assume X=Y and we will classify them from the point of view of various "properties of relations". Next, we will introduce one type of relations between elements from (potentially) different set, namely functions.

Properties of relations

Definition.

Let R be a relation on a set X, i.e. $R \subseteq X \times X$. We say that:

1. R is *reflexive* iff $(\forall x \in X)xRx$
2. R is *symmetric* iff $(\forall x, y \in X)(xRy \Rightarrow yRx)$
3. R is *transitive* iff $(\forall x, y, z \in X)[(xRy \wedge yRz) \Rightarrow xRz]$
4. *R is antisymmetric* iff $(\forall x, y \in X)[(xRy \wedge yRx) \Rightarrow x = \mathrm{y}]$
5. R is *total* iff $(\forall x, y \in X)(xRy \vee yRx)$

Occasionally we will use the idea of the *inverse relation* in the following sense:

If R is a relation on X then the inverse relation $R^{-1} = \{(p, q) \in X \times X | (q, p) \in R\}$

Examples.

1. Reflexivity. EQ is reflexive, $\leq$ is reflexive, $\subseteq$ is reflexive.
   The *congruence modulo n* relation on $\mathbb{Z}$ where n is a natural number: $a \equiv b(mod\ n)$ iff $n|(a-b)$. Clearly, for every n congruence mod n is reflexive.
   The symbol $p(mod\ n)$ is also used to denote the remainder of the division of p by n. Hence, we can say
   $a \equiv b(mod\ n)$ iff $(a-b)mod\ n = 0$.
   Is divisibility relation | reflexive? It depends. It is in $\mathbb{N}$ it is not in $\mathbb{Z}$ because 0 is not divisible by 0. Of course divisibility on $\mathbb{N}$ and divisibility on $\mathbb{Z}$ are two different relations. Remember: a relations is a set!
   Relations $<$, $\neq$, xRy iff $y=x^2$ are not reflexive.

   Fact. R is reflexive iff $EQ \subseteq R$. This implies that the inverse of a reflexive relation is reflexive as well.

2. Symmetricity. EQ is symmetric, $\leq$ is not, $\subseteq$ is not, < is not.
Congruence mod n is symmetric for every n (if n divides $a - b$ then n divides $b - a$).
Divisibility on $\mathbb{N}$ is not symmetric (2|4 but $\neg 4|2$).
$\neq$ (meaning the relation of "being different from" is symmetric.

Fact. R is symmetric iff $R \subseteq R^{-1}$.
In fact, we can say that R is symmetric iff $R = R^{-1}$.
Proof. ($\Rightarrow$)Suppose R is symmetric and aRb. Then bRa, i.e. $aR^{-1}b$. This proves $R \subseteq R^{-1}$. Now, suppose $pR^{-1}q$. This means qRp. Since R is symmetric, we obtain pRq, i.e. $R^{-1} \subseteq$ R.
The other implication ($\Leftarrow$) is just as easy.

3. Transitivity. EQ, $<$, $\leq$, $\geq$, $\subseteq$ are transitive. $\neq$ is not.
   Is $\equiv_n$ (congruence mod n) transitive? Suppose $a \equiv_n b$ and $b \equiv_n c$. This translates into $n|a-b$ and $n|b-c$. In other words, $a-b=kn$ and $b-c=ln$ for some k and l. Adding the last two equations we get $a-b+b-c=ln+kn$. Simplifying this we get $a-c=n(l+k)$, which means n divides a-c, i.e. $a \equiv_n c$.
   Divisibility is transitive.
   Is the relation "being friendly with" transitive? Is "a friend of my friend" a friend of mine? Suppose you meet another boyfriend of your girlfriend … Oops, awkward.

Transitivity is hard to express in terms of set operations on relations. Probably because it says something about a pair composed of the first element of one related pair and the second element of another…

4. Antisymmetricity. EQ is antisymmetric. So are $\leq$, $\geq$ and $\subseteq$. What about < on $\mathbb{R}$? Suppose a<b and at the same time b<a. Impossible? Good, it means that the statement (a<b)∧ (b<a) is FALSE. But every implication with false left-hand side is true hence, < is antisymmetric.
   Is $\equiv_n$ antisymmetric? Obviously not: $3 \equiv_2 5$ and $5 \equiv_2 3$ but 3 is not equal 5.
   Divisibility is antisymmetric.

Fact.

R is antisymmetric iff $R \cap R^{-1} \subseteq EQ$

5. Totality. EQ is total. So are $\leq$ and $\geq$.
$\subseteq$ is not (unless $|X| \leq 1$), because if you have two different elements a and b in X then $\{a\}$ and $\{b\}$ are not related neither one way nor the other.
$\equiv_n$, divisibility are not total (in general). But divisibility is a total relation on, for example, the set $\{2^n | n \in \mathbb{N} \cup \{0\}\}$.

Fact.

R is total iff $R \cup R^{-1} = X \times X$

Equivalence relations

Definition.

A relation R on a set X is called *an equivalence relation* iff R is reflexive, symmetric and transitive.

This notion was constructed on the basis of the EQuality relation. We tend to think about equivalent elements as "indistinguishable from some point of view".

Examples.

1. EQ,
2. "equal size" or *equipotency* relation (set A is related to B iff |A|=|B|) on the set of finite subsets of X.
3. "parallel" relation on the set of all lines on the plane,
4. "equal modulus" in $\mathbb{C}$ ($z_1 R z_2$ iff $|z_1| = |z_2|$)
5. congruence mod n on $\mathbb{Z}$
6. "concentric" on the set of all balls in $\mathbb{R}^3$ (or all spheres).
7. the full relation $R = X \times X$

Negative examples.

- $<$ (not reflexive, not symmetric),
- $\neq$ (symmetric but neither reflexive nor transitive)
- $\leq$ and $\subseteq$ (reflexive and transitive but not symmetric)
- "friends". Assuming everybody are friendly with themselves it is reflexive and (hopefully) symmetric but not transitive.
- "perpendicular" on the set of all lines on the plane (symmetric but neither reflexive nor transitive). Notice that "perpendicular or parallel" is an equivalence.
- the empty relation on a nonempty set X

With an equivalence relation R defined on X, related elements of X are considered *equivalent* or indistinguishable from some point of view. This often (in fact *always*) means that related (equivalent) elements share some parameter or property. Sometimes the property is obvious, sometimes less so.

For example:

Equipotent sets share the size (the number of elements).

Parallel lines share direction.

What property or parameter do congruent integers share?

Suppose $p \equiv_n t$ i.e. $n|p-t$. From the remainder lemma for integers, there exist unique q_p, q_t, r_p and r_t such that $p = nq_p + r_p$, $t = nq_t + r_t$ and $0 \leq r_p, r_t \leq n-1$. Then $p - t = n(q_p - q_t) + r_p - r_t$ is divisible by *n*. Since $n(q_p - q_t)$ is divisible by *n*, $r_p - r_t$ must be as well. From $0 \leq r_p, r_t \leq n-1$ we obtain that $-n+1 \leq r_p - r_t \leq n-1$. The only number in this interval divisible by *n* is 0 hence, $r_p = r_t$. In other words, numbers congruent mod *n* share the remainder from the division by *n*. We can also say $p \equiv_n t$ iff $p(mod\ n) = t(mod\ n)$.