EIDMA
Lecture 5

Functions

FUNCTIONS

Definition.

A relation $f \subseteq X \times Y$ is called a *function from X into Y* iff

(1) $(\forall x \in X)(\exists y \in Y)(x, y) \in f$ (or xfy or $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$)

(2) $(\forall x \in X)(\forall z, y \in Y)[(xfy \wedge xfz) \Rightarrow y = z]$

or, equivalently, $[(f(x) = y \wedge f(x) = z) \Rightarrow y = z]$

x is called an *argument* of *f* and *y* – the *value* assigned to *x* by *f*.

Condition (1) guarantees that every argument is assigned a value and condition (2) that the value is unique.

If f is a function from X into Y, we usually write $f: X \to Y$.

Commonly used notation for xfy and $(x,y) \in f$ is $f(x) = y$.

X is called the *domain*, and Y – the *range* or *codomain* of f.

All arguments belong to the domain, all values to the range.

Think of functions as ordered triples (X,Y,f).

What does it mean that two functions, say $f: X \to Y$ and $g: Z \to T$ are equal? According to the last remark this means that the ordered triples (X, Y, f) and (Z, T, g) are equal. Hence $x^2|_{\mathbb{R}}$ and $x^2|_{<0;\infty)}$ should be considered different because they have different domains. The function is not just the rule of assignment, it is all three things, the domain, the range **and** the rule of assignment (which is a subset of $X \times Y$, really).

Comprehension. Which of the following relations are and which are not functions:

1. $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$
2. $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = y\}$
3. $f = \{(x, y) \in R \times R \mid x = y^2\}$
4. $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = y^2\}$
5. $f = \{(x, y) \in X \times X \mid x = y\}$, X denotes any set.
6. f^{-1} when $f: X \to Y$ is a function. Here, f^{-1} denote the inverse relation to the relation f. It may be confusing because all relations are invertible in the sense that for every relation we can create the inverse relation. The question is, is the inverse relation to a function also a function.

The set of all functions from X into Y is denoted by Y^X. One reason why we use this notation is that the formula for the number of functions from a finite set X into a finite Y looks nice:

$$|Y^X| = |Y|^{|X|}$$

Comprehension question. Prove the above formula.

ALGEBRA OF FUNCTIONS

Definition.(Composition of functions)

Let $f: X \to Y$ and g:$Y \to Z$. The *composition* of g and f is the function $g \circ f$ such that for every x, $(g \circ f)(x) = g\big(f(x)\big)$.

Notice: Values of f must belong to the domain of g, otherwise composition makes no sense.

Fact. Function composition is associative i.e. for every three functions f, g, h $(f \circ g) \circ h = f \circ (g \circ h)$.

The question of commutativity is more complicated because it is possible that $g \circ f$ exists while $f \circ g$ does not.

Fact. Composition of functions is not, in general, commutative even if both $g \circ f$ and $g \circ f$ exist.

Example.

$f(x) = 2x,\ g(x) = 2 + x$. Then

$(f \circ g)(x) = f(g(x)) = f(2 + x) = 2(2 + x) = 4 + 2x$

$(g \circ f)(x) = g(f(x)) = 2 + f(x) = 2 + 2x.$

Are $f \circ g$ and $g \circ f$ equal or not? They are different not because they *look* different but because to some *x*-*s* (e.g. to *x*=0) they assign different values.

Fact.

The pair $(X^X, \circ)$ is an associative algebra with an identity element.

Proof.

Composition of two functions mapping X into X is also a function mapping X into X.

The identity element is the function id_X such that for every $x \in X$, $id_X(x) = x$. This function clearly satisfies the condition $id_X \circ f = f \circ id_X = f$ for every function f.

Fact.

If the inverse relation f^{-1} for a function $f: X \to Y$ is also a function, from Y into X, then it is called the *inverse* (*function*) of *f*. Then

$$f \circ f^{-1} = id_Y \text{ and } f^{-1} \circ f = id_X$$

Examples.

- $\log_2(x)$ is the inverse function of 2^x (and vice versa)
- $\sqrt{x}$ is the inverse to x^2 (x^2 restricted to $\mathbb{R}^+$)
- arcsin x is the inverse to sin x (if sine is considered a function from $[\frac{-\pi}{2}; \frac{\pi}{2}]$ into $[-1; 1]$)

Comprehension.

f:X→Y. g is the inverse function for f iff

$$(\forall x \in X)(\forall y \in Y)(f(x)=y \Leftrightarrow g(y)=x)$$

Image and inverse image

Definition.
Let $f: X \to Y$ and $A \subseteq X$. The set $f(A) = \{f(x) | x \in A\}$ is called *the image* of A by f. In other words, the image of a set A is the set of values of f assigned to elements of A.

Comprehension.
Find $\sin(<0;\pi>)$, $\cos(<0;\pi>)$, $\sin(<-\pi;0>)$, $\sin(\mathbb{R})$

Definition.
Let $f: X \to Y$ and $B \subseteq Y$. The set $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ is called *the inverse image* of B by *f*.

In other words, the inverse image of a set B is the subset of the domain *X* consisting of all those elements, for whom *f* assigns values belonging to B.

Note. The symbol $f^{-1}(B)$ makes sense even if *f* is not invertible. Some authors prefer notation $f[A]$ to $f(A)$ and $f^{-1}[B]$ to $f^{-1}(B)$. to avoid ambiguity: $f^{-1}(B)$ might be understood as "the image of *B* by the function f^{-1}" and $f(A)$ may sometimes denote the value of f for an argument denoted by *A*.

Example. Let P be the function which assigns prices to guitars in a music store. If you ask the attendant to show you guitars in the 1500 to 2000 price range, you are in fact asking him to construct $P^{-1}(<1500;2000>)$.

Comprehension.
Find $\sin^{-1}(<0;\pi>)$, $\cos^{-1}(<0;\pi>)$, $\sin^{-1}(<-\pi;0>)$, $\sin^{-1}(\mathbb{R})$.

Example.

Let $f: X \to Y$. Is it true that for every $A \subseteq X$, $f^{-1}\big(f(A)\big) = A$?

This question is equivalent to

"Is $[A \subseteq f^{-1}\big(f(A)\big) \wedge f^{-1}\big(f(A)\big) \subseteq A]$ true for every $A \subseteq X$?

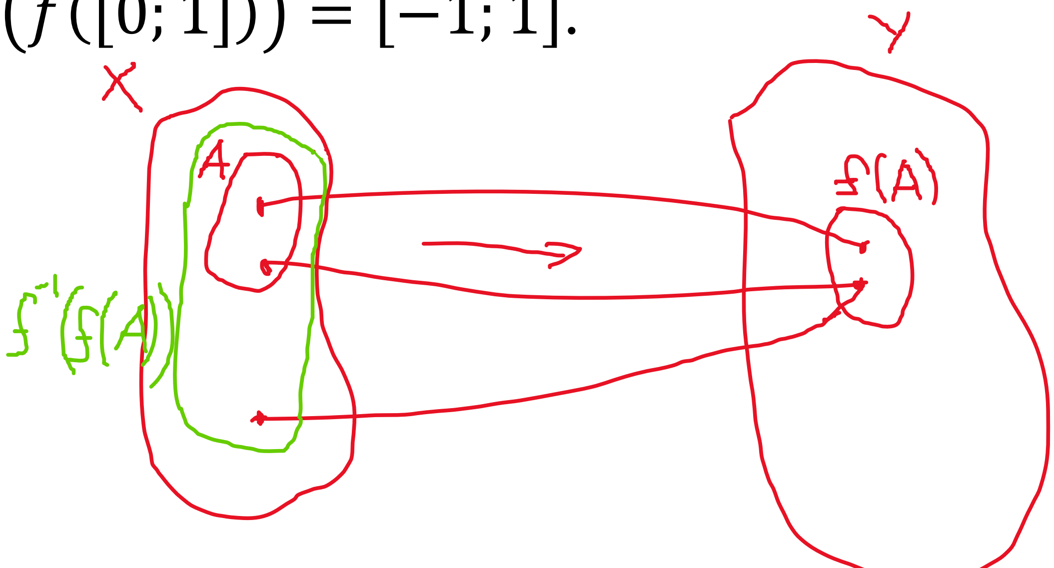
Let $a \in$ A. What (the hell) does $a \in f^{-1}\big(f(A)\big)$ mean? We look-up the definition and it turns out that $f^{-1}\big(f(A)\big) = \{x \in X | f(x) \in f(A)\}$. Clearly, $f(a) \in f(A)$, i.e., $a \in f^{-1}\big(f(A)\big)$ and $A \subseteq f^{-1}\big(f(A)\big)$.

Now, we must attempt to prove $f^{-1}\big(f(A)\big) \subseteq A$. Take some $b \in f^{-1}\big(f(A)\big)$. As before, it means that $b \in f(A)$, so for some $c \in A, f(c) = f(b)$. Does this mean that $b \in A$? Not necessarily, some elements outside A might be assigned the same values as some elements from A. E.g. if $f(x) = x^2$ then $f^{-1}\big(f([0;1])\big) = [-1;1]$.

Example – cont.

Let $f: X \rightarrow Y$. Is it true that for every $A \subseteq X$, $f^{-1}\big(f(A)\big) = A$?

Now, we must attempt to prove $f^{-1}\big(f(A)\big) \subseteq A$. Take some element $b \in f^{-1}\big(f(A)\big)$. As before, it means that $f(b) \in f(A)$, so for some $c \in A, f(c) = f(b)$. Does this mean that $b \in A$? Not necessarily, some elements outside A might be assigned the same values as some elements from A. E.g. if $f(x) = x^2$ then $f^{-1}\big(f([0;1])\big) = [-1;1]$.



This could not happen if f were 1-1.

Example.

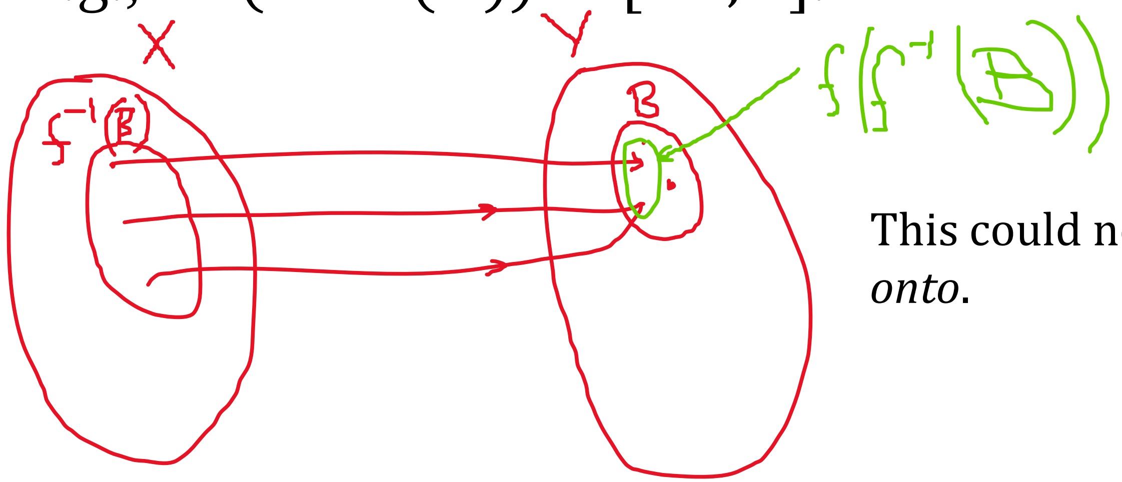
Let $f: X \rightarrow Y$. Is it true that for every $B \subseteq Y$, $f\left(f^{-1}(B)\right) = B$?

This question is equivalent to

"Is $[f\left(f^{-1}(B)\right) \subseteq B \wedge B \subseteq f\left(f^{-1}(B)\right)]$ true for every $B \subseteq Y$?

Let $p \in f\left(f^{-1}(B)\right)$. This means $p = f(q)$ for some $q \in f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ hence, $p = f(q) \in B$.

Now, take $q \in B$. Does this mean that $q = f(r)$ for some $r \in f^{-1}(B)$? Not necessarily, some elements of B may not be values of f at all. E.g., $\sin(\sin^{-1}(\mathbb{R})) = [-1; 1]$.



This could not happen if f were *onto*.

Definition.

A function $f: X \to Y$ is said to be *injective* (or a 1-1 *function*) iff

$$(\forall a, b \in X)(a \neq b \ \Rightarrow f(a) \neq f(b))$$

Fact.

A function *f* is 1-1 iff $(\forall a, b \in X)(f(a) = f(b) \Rightarrow a = b))$

Fact.

A function *f* is 1-1 iff $(\forall y \in Y)|f^{-1}(\{y\})| \leq 1$

Comprehension.

Which functions are 1-1:

person $\rightarrow$ father, car $\rightarrow$ price,

complex number z $\rightarrow$ Re z,   z $\rightarrow |z|$,

real number x $\rightarrow \sqrt[3]{x}$, x $\rightarrow$ sin x, x $\rightarrow$ decimal expansion of x

Definition. A function $f: X \to Y$ is said to be *surjective* (or an '*onto*' *function*) iff

$$(\forall b \in Y)(\exists a \in X) f(a) = b$$

Fact.

A function f is a surjection iff $f(X) = Y$.

Definition.

A function which is both 1-1 and onto is called a *bijection*

The properties of 1-1, onto and bijection depend on the choice of the domain and the range. This is one more reason why we should think of a function as a triple (Domain, Range, Relation) rather than the relation alone (or the rule of assignment).

Example.

$f(x) = x^2$ is:

- surjective and not injective if considered a function from $\mathbb{R}$ into $\mathbb{R}^+ \cup \{0\}$
- not surjective and not injective if considered a function from $\mathbb{R}$ into $\mathbb{R}$.
- bijection if considered a function from $\mathbb{R}^+ \cup \{0\}$ into $\mathbb{R}^+ \cup \{0\}$.

It is more consistent to consider them different functions rather than one capricious function which sometimes is this and sometimes that.

CONTEXT IS EVERYTHING