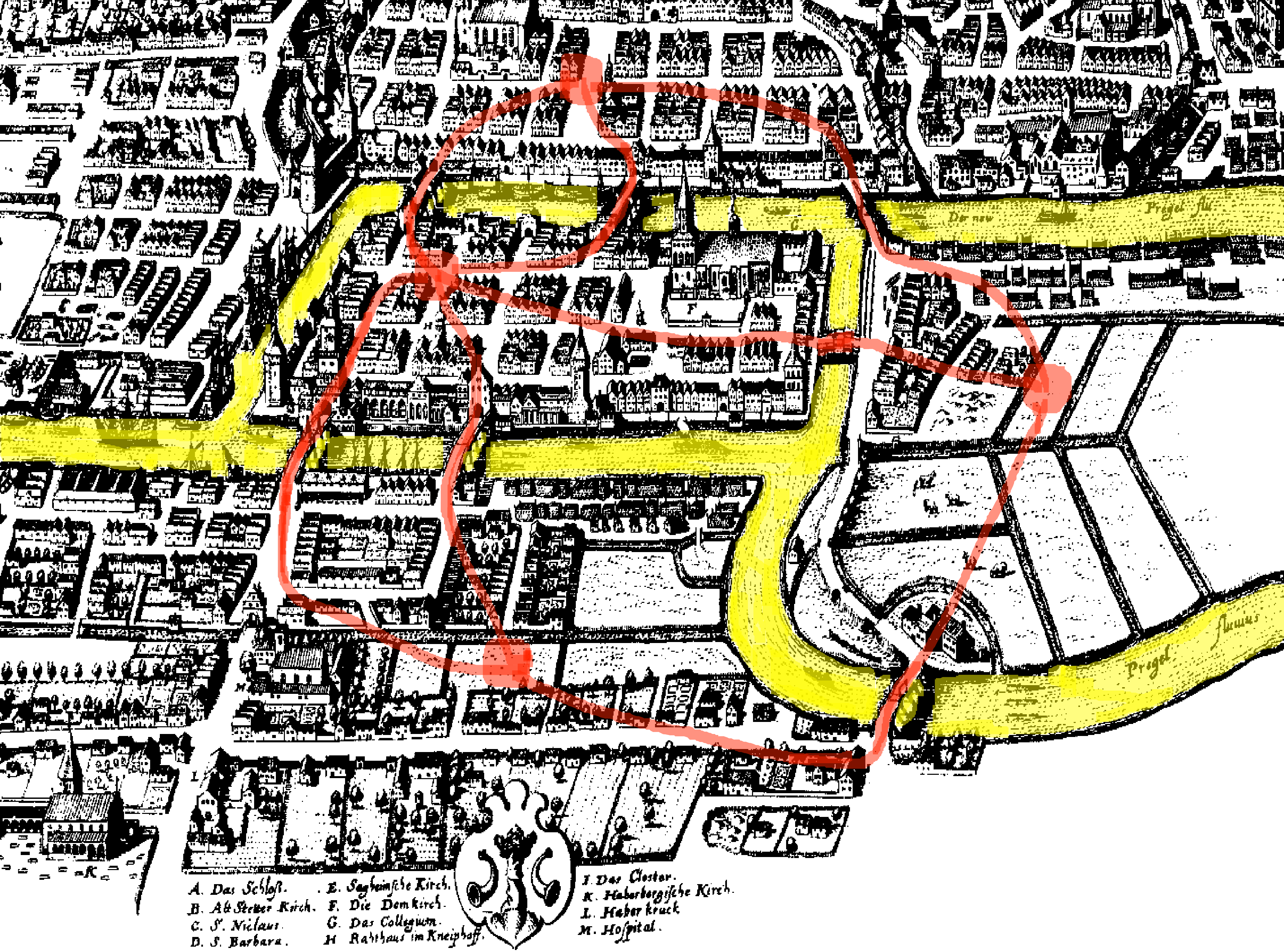
EIDMA

Lecture 13

- Eulerian graphs,
- Hamiltonian graphs



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A. Das Schloß.
B. Alt Stetter Kirch.
C. S. Niclaus.
D. S. Barbara.
E. Sagheimsche Kirch.
F. Die Domkirch.
G. Das Collegium.
H Rathhaus im Kneiphoff.
I. Das Closter.
K. Haberbergische Kirch.
L. Haber kruck.
M. Hospital.

Definition.
An *Eulerian cycle* in a graph G is a cycle which passes through all edges and vertices of G.

Definition.
A graph is said to be *Eulerian* iff it has an Eulerian cycle.

Fact. (obvious)
Every Eulerian graph G is connected, has at least three vertices and each vertex of G has an even degree.

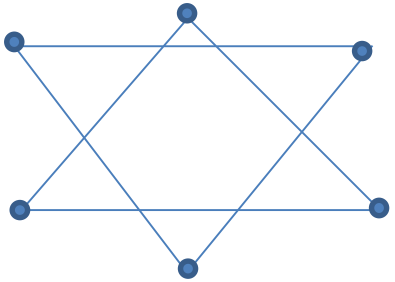
Definition.
Each maximal connected subgraph of a graph G is called a *component* of G. (Some texts use the term *connected component* which I consider an overkill.)

Definition.
The *distance* between vertices x and y in G, $dist_G(x,y)$, is the length of a shortest x-y path in G or ∞ if no such path exists.

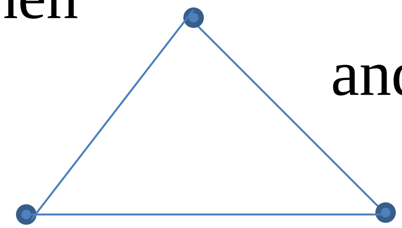
Fact. (obvious)
Components of G are subgraphs induced by equivalence classes of the following relation on V(G): $v\mathrm{R}u$ iff $dist_G(x,y)$ is finite.

Example.

Let G be



Then



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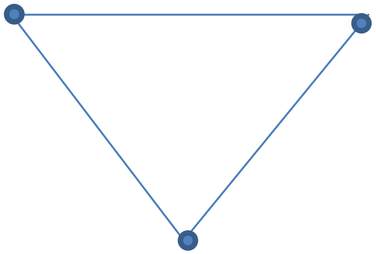
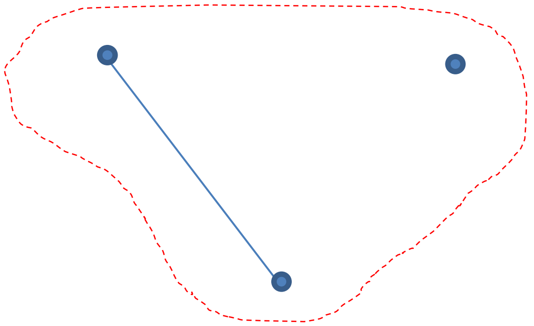
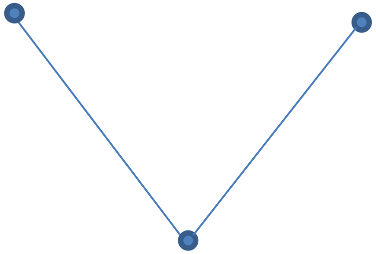


are components of G

while



and



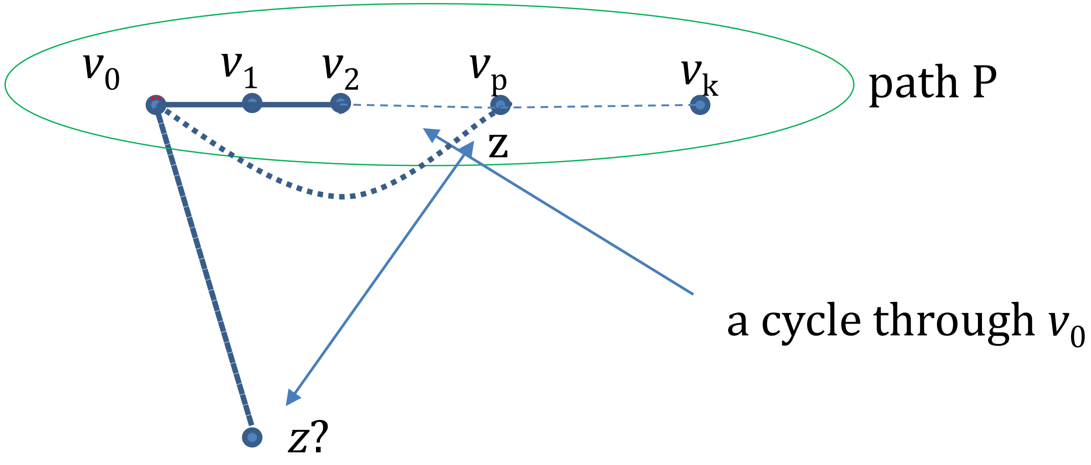
are not

Lemma.

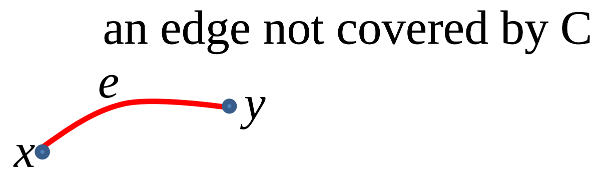
If for every vertex v of G $\deg(v) \geq 2$ then G has a cycle.

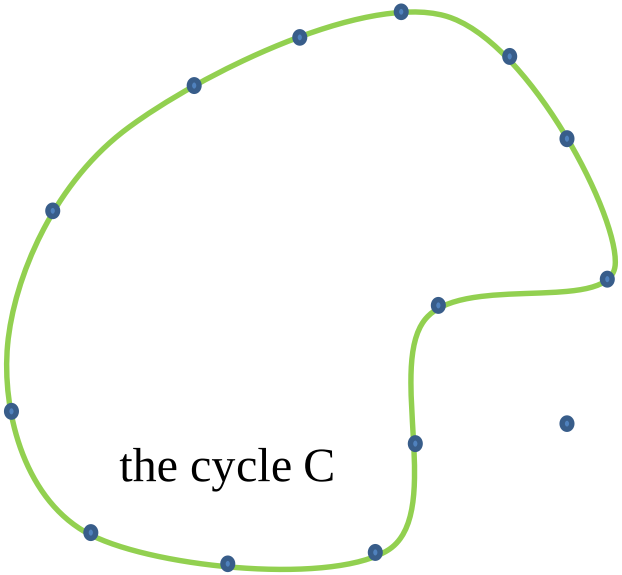
Proof.

Let P = $(v_0, v_1, \ldots, v_k)$ be a longest simple path in G. Since $\deg(v_0) \geq 2$, v_0 has a neighbor z, different from v_1. If $z \notin$ P then $(z, v_0, v_1, \ldots, v_k)$ is a simple path longer than P, which is not possible, so $z = v_p$ for some $1 < p \leq k$. Then $(v_0, v_1, \ldots, v_p, v_0)$ is a cycle in G passing through v_0.

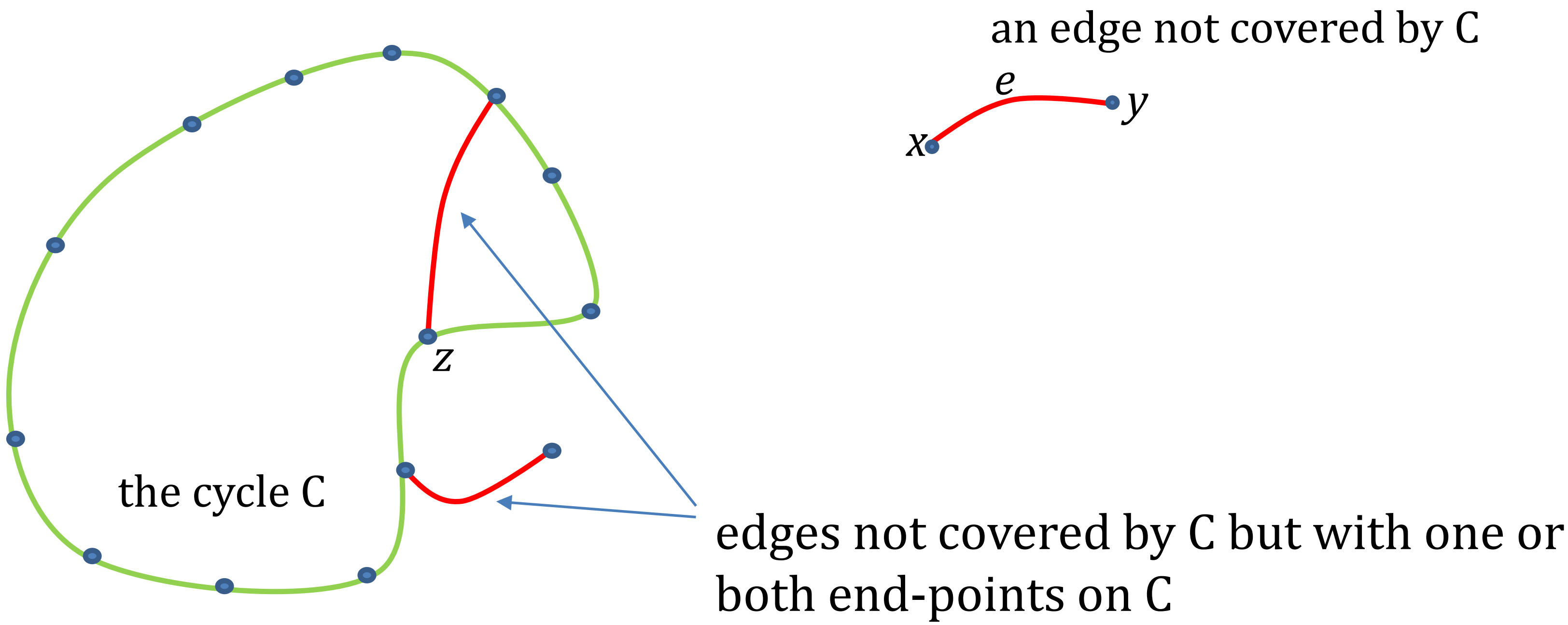


Theorem. (Euler, 1736)
A graph G is Eulerian iff G has more than 1 vertex, G is connected and every vertex of G has an even degree.
Proof. ("$\Rightarrow$" is obvious, we only do "$\Leftarrow$", by induction on $n=|E|$)
1. A connected graph G with more than one vertex and all even degrees must have at least 3 vertices and all degrees at least equal 2. So, by the last lemma G, has a cycle. We begin with $n=3$. Th only graph on 3 edges satisfying our conditions is K_3 which is obviously Eulerian.
2. Suppose G has n edges, $n>3$, G satisfies our conditions and for every graph on fewer edges the theorem is true. There are some cycles in G due to the Lemma. Denote by C a longest cycle in G. If C is an Eulerian cycle we are done. Suppose to the contrary that C does not cover all edges. Let $e = \{x,y\}$ be an edge of G not covered by C.

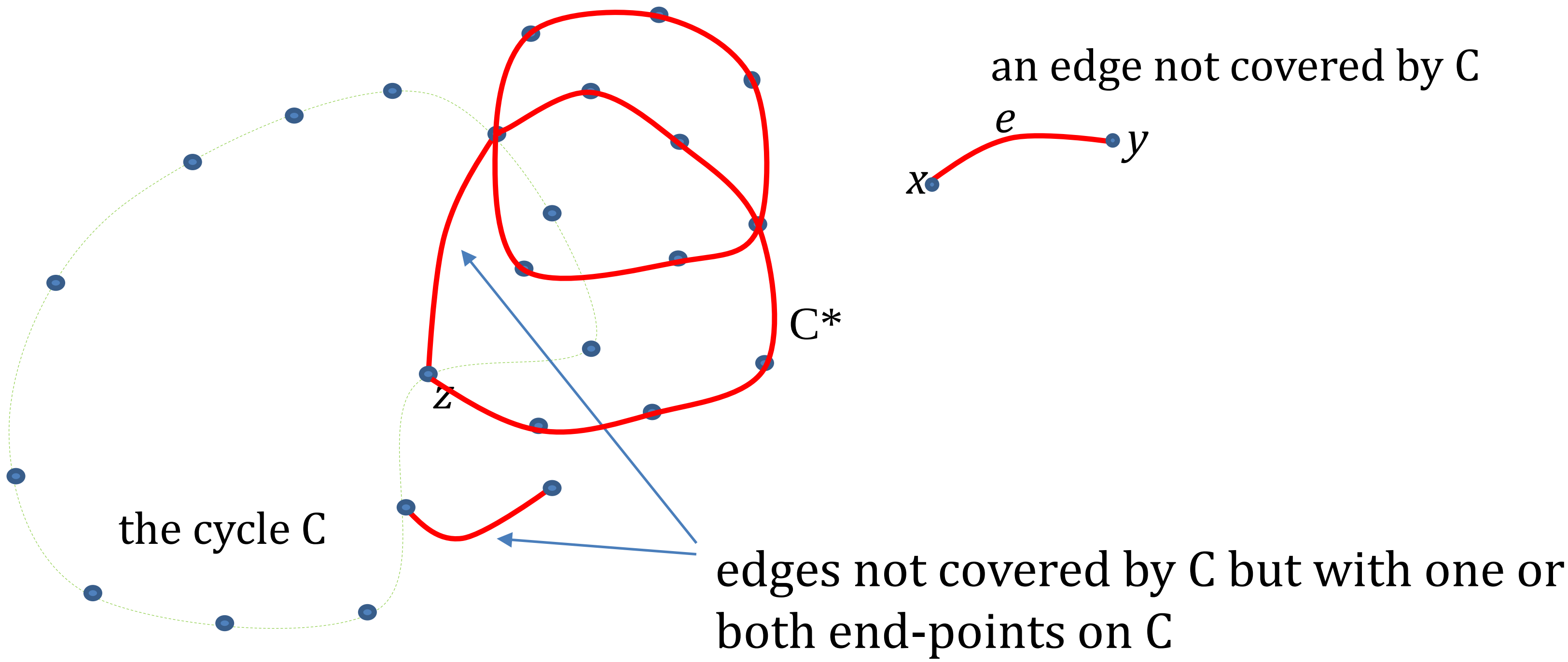




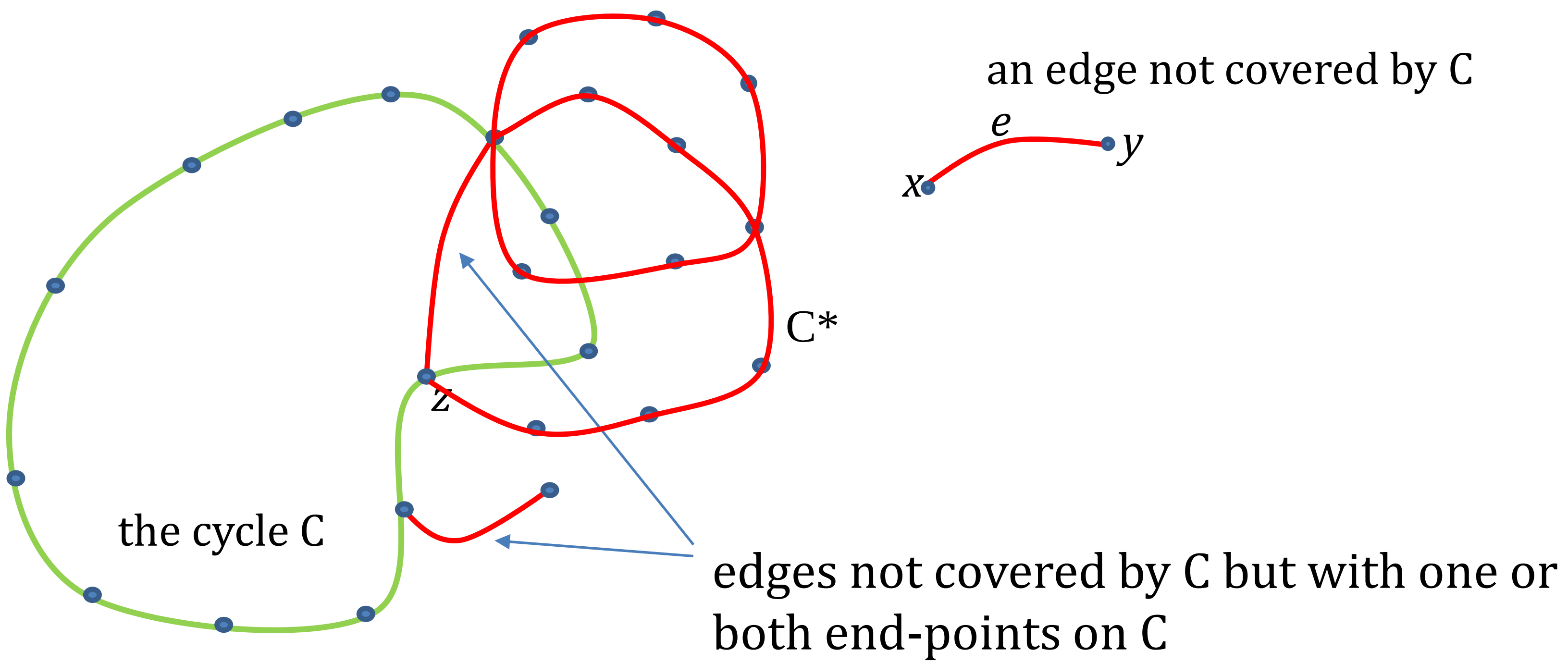
If there is an edge not used by C then there must exist such an egde with at least one end-point in C.
Otherwise, consider the set X consisting of the endpoints of all edges not belonging to C. Then X and V(C) are disjoint, nonempty and $X \cup V(C) = V(G)$, i.e. X and V(C) form a partition of V(G) and there are no edges between vertices of X and those of V(C) – contrary to the partition theorem about connected graphs.



Consider the subgraph G* = G – E(C) obtained by the removal from G of all edges of C. In G* every vertex has an even degree, some perhaps 0. But the endpoints of the red edges have degrees at least 2.



The component H_z of G* containing *z* has fewer edges than G does so, by our induction hypothesis H_z is Eulerian. We denote its Eulerian cycle by C*.



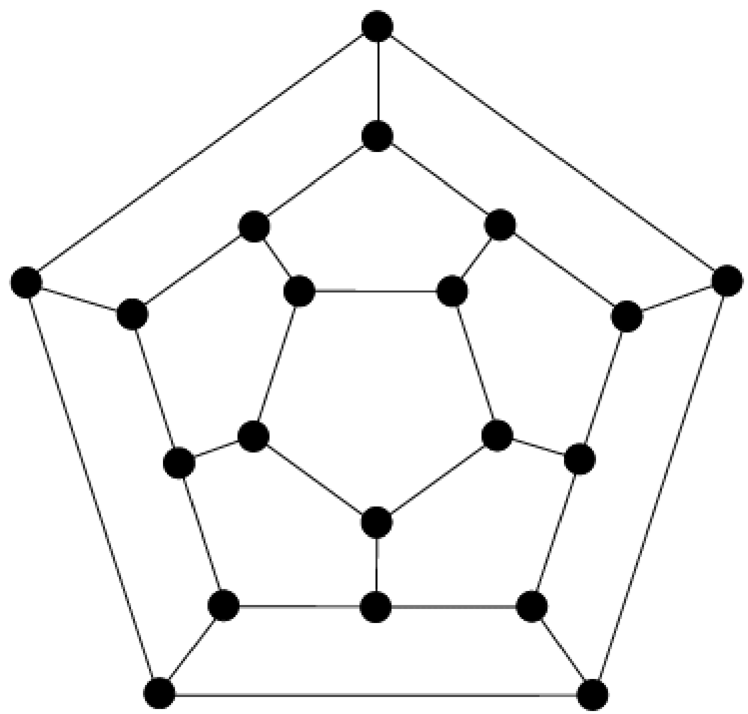
Starting from z we travel along the red cycle C*. When we cover whole C* and get back to z we travel the green cycle C thus constructing a cycle longer than C – contrary to our claim that C is a longest cycle in G.

The proof can easily be transformed into an algorithm for finding an Eulerian cycle in a graph G satisfying Euler's conditons:

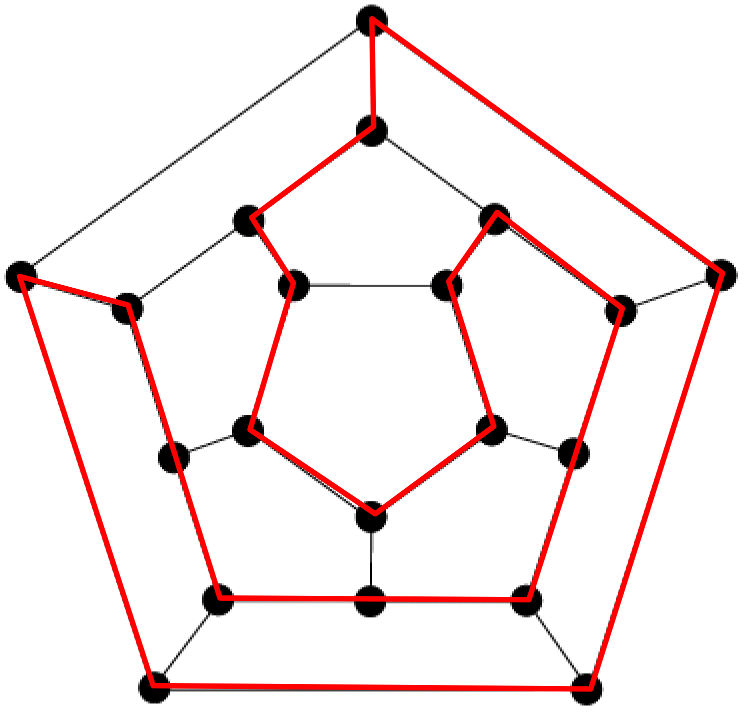
- Start from any vertex x_0 and construct a cycle passing through x_0, for example by random walk. Call the cycle C_0. If C_0 covers all edges and vertices - STOP
- Else: create G_1 by erasing from G all edges of C_0, find a vertex x_1 of C_0 with a positive degree in G_1 and construct in G_1 a cycle C_1 through x_1
- Repeat until cycle C_k covers all remaining edges.
- Start from x_0 and at every intersection go along an edge belonging to a C_p with the largest subscript p.

Hamiltonian graphs

The legend of Sir Wiliam R. Hamilton and his game



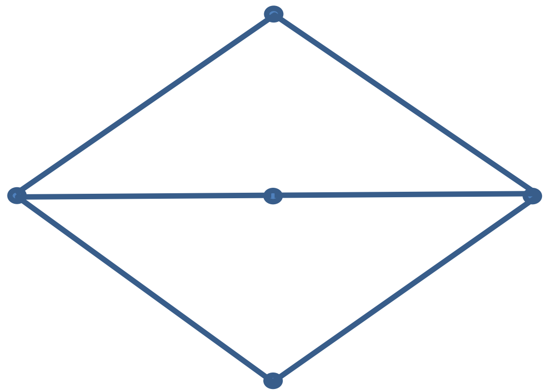
Hamiltonian cycle in Hamilton's game graph

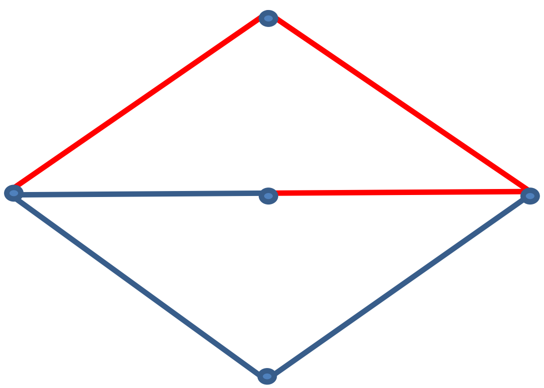


Definition.

A graph is said to be *Hamiltonian* iff it contains a simple spanning cycle, called a Hamiltonian cycle.

Obviously, every Hamiltonian graph is connected and has all vertices of degree at least 2.

This example  shows that it is not enough.



FAQ.

The definitions of Hamiltonian and Eulerian properties seem pretty similar. Is there a connections between the two?

Answer. No. On the contrary. Euler's theorem characterizes Eulerian graphs. There is no such theorem for Hamiltonian graphs. Not yet, anyway.

Comprehension.

Find examples showing that the two properties are logically independent (meaning having (or not having) one does not imply having (or not having the other)).

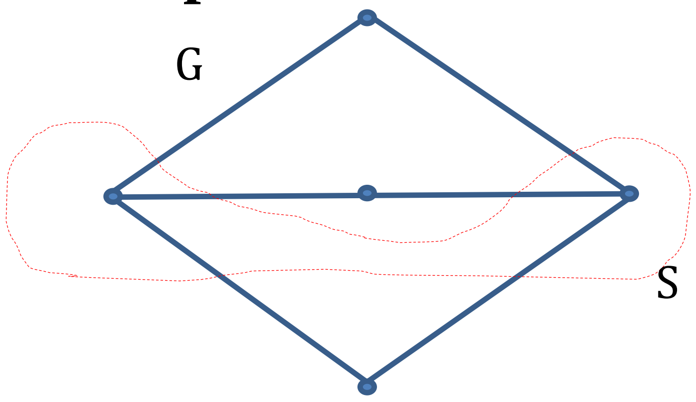
NECESSARY CONDITIONS.

Definition.

Let G =(V,E) be a graph and let S be a subset of V. Then G-S is the subgraph of G obtained by removing form G all vertices from S (together their incident edges, of course).

In other words, G-S is the subgraph of G induced by V\S.

Example

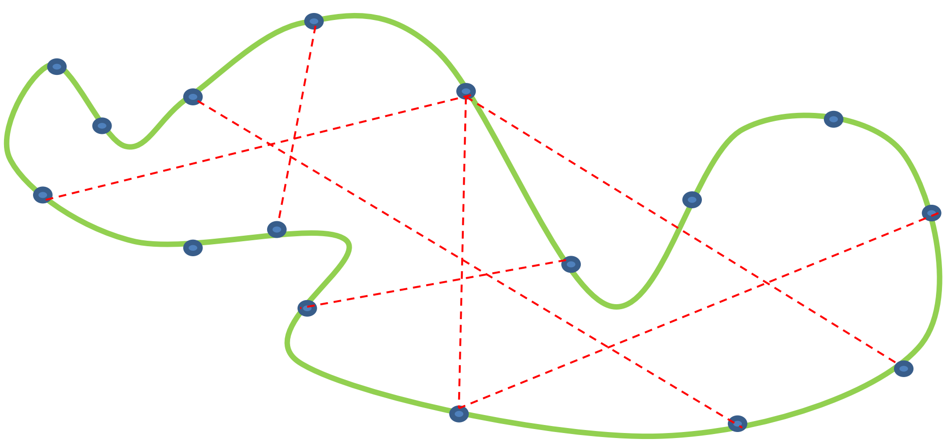


Theorem.

If G=(V,E) is Hamiltonian then for every *k* and for every *k*-element subset S of V the number of components in G-S is not greater than *k*.

Proof.

Every Hamiltonian graph looks like this (green edges)



plus, possibly, some extra edges (red dotted ones). Removing *k* vertices from the green cycle splits the cycle into at most *k* paths. Some of those paths may be connected by red edges making the number of components, if anything, smaller.