SOLUTIONS 2015-02-09

1. Prove that in every nontrivial graph there exist two vertices of the same degree. (Hint: *Pigeonhole principle*)
   Solution. Let n be the number of vertices, $n>1$. The smallest possible degree of a vertex is 0 (for vertices not adjacent to any other vertices), the largest is n-1 (for vertices adjacent to every vertex). But both cannot happen in a graph (if there is a vertex adjacent to all vertices there there none adjacent to none). Hence, all degrees are between 0 and n-2 or all are between 1 and n-1. In both cases the number of available degrees is at most n-1 so, by pigeonhole principle, however you distribute the degrees among the vertices, at least 2 vertices will get the same degree.

2. Consider a relation R on $\mathbf{R}^2 - \{(0,0)\}$ such that $(a,b)R(c,d)$ iff $bc=ad$. Is R an equivalence relation? If the answer YES, find its equivalence classes.
   Solution. This is a relation between pairs of numbers who are not BOTH equal to zero (or between points on the plane represented by these pairs, except the origin).
   <u>Reflexivity</u> means that every point is related to itself, which means, for every two numbers a and b, $(a,b)\mathrm{R}(a,b)$, i.e. $ab=ba$, which is obvious from algebra.
   <u>Symmetricity</u> means that $(\forall a,b,c,d)$ $(a,b)\mathrm{R}(c,d) \Rightarrow (c,d)\mathrm{R}(a,b)$. By definition of R this is equivalent to $(\forall a,b,c,d)$ $ad=bc \Rightarrow cb=da$
   <u>Transitivity</u> is a tiny bit harder. By definition of R it means $(\forall a,b,c,d,e,f)$ $[(ad=bc \;\&\; cf=de) \Rightarrow af=be]$. Since $(d,c)\neq(0,0)$ either $d\neq 0$ or $c\neq 0$ (or both). Suppose $d\neq 0$ (the case $c\neq 0$ is similar). Dividing the first two equalities by d we get $a=\frac{bc}{d}$ and $e=\frac{cf}{d}$. Multiplying both expressions by f and b, respectively, we get $af=\frac{bcf}{d}=\frac{cfb}{d}=be$, which proves that R is reflexive.
   Clearly, for every point (a,b) its equivalence class is the line passing through (a,b) and $(0,0)$, without the point $(0,0)$.

3. Quote and prove the Newton formula for the number of k-element subsets of an n-element set.
   Solution. The formula, $\frac{n!}{(n-k)!\,k!}$, and the proof were presented during the course. Look through your notes.

4. There are n chairs in a row. In how many ways can a teacher sit k students on these chairs so that no 2 students sit next to each other (and obviously no 2 students sit on 1 chair)?
5. How many permutations of letters of the word COMBINATORICS do not have identical consecutive letters?

Tasks 4 and 5 are standard combinatorial exercises.