SOLUTIONS EIDMA E1 2015-06-17

1. Prove that in every graph G on p vertices if every vertex has degree at least $\frac{p}{2}$ then G is connected.

   Solution 1(By first principles). We must prove that for every two vertices u and v there is a u-v path in G. If there exists a vertex z adjacent to both u and v then (u,z,v) is a u-v path. Otherwise the sets N(u) and N(v) are disjoint (N(x) denotes the set of all neighbors of x). Then $p \geq |N(u) \cup N(v)| = |N(u)| + |N(v)| = \deg u + \deg v \geq p$. This means that every vertex of G is a neighbor of u or v. Since u is not a neighbor of u we get that u and v are adjacent.

   Solution 2.(By the Partition Theorem) Suppose that G is not connected. Then there exists a partition of V into A and B such that no vertex of A and no vertex of B are adjacent. Without loss of generality we may assume that $|A| \leq \frac{p}{2}$. Then for every $x \in A$ $\deg x \leq \frac{p}{2} - 1 < \frac{p}{2}$. Contradiction.

2. Consider a relation R on $\mathbf{R}^2 - \{(0,0)\}$ such that $(a,b)R(c,d)$ iff $bc=ad$. Is R an equivalence relation? If the answer YES, find its equivalence classes.
   Solution. This is a relation between pairs of numbers who are not BOTH equal to zero (or between points on the plane represented by these pairs, except the origin).
   <u>Reflexivity</u> means that every point is related to itself, which means, for every two numbers a and b, $(a,b)R(a,b)$, i.e. $ab=ba$, which is obvious from algebra.
   <u>Symmetricity</u> means that $(\forall a,b,c,d)$ $(a,b)R(c,d) \Rightarrow (c,d)R(a,b)$. By definition of R this is equivalent to $(\forall a,b,c,d)$ $ad=bc \Rightarrow cb=da$, which is obvious
   <u>Transitivity</u> is a tiny bit harder. By definition of R it means $(\forall a,b,c,d,e,f)$ $[(ad=bc \; \& \; cf=de) \Rightarrow af=be]$. Since $(d,c)\neq(0,0)$ either $d\neq 0$ or $c\neq 0$ (or both). Suppose $d\neq 0$. Dividing the first two equalities by d we get $a = \frac{bc}{d}$ and $e = \frac{cf}{d}$. Multiplying both expressions by f and b, respectively, we get $af = \frac{bcf}{d} = \frac{cfb}{d} = be$. If d is equal to 0, c is not and we multiply the first equality by $\frac{e}{c}$ and the second by $\frac{a}{c}$, getting $be = \frac{ade}{c} = \frac{dea}{c} = fa$, which proves that R is transitive.
   Clearly, for every point (a,b) its equivalence class is the line passing through (a,b) and $(0,0)$, without the point $(0,0)$.

3. Quote and prove the formula for the number of k-element subsets of an n-element set.
   Solution. The formula, $\frac{n!}{(n-k)!k!}$, and the proof were presented during the course. Look through your notes.

4. Find the number of n-long sequences of A,B,C and D not containing A or B or C.
   Solution. Let X(A) be the set of all n-long sequences not containing A, similarly X(B) and X(C). What we need is $k = |X(A) \cup X(B) \cup X(C)|$. By inclusion-exclusion principle $k = |X(A)| + |X(B)| + |X(C)| - |X(A) \cap X(B)| - |X(A) \cap X(C)| - |X(C) \cap X(B)| + |X(A) \cap X(B) \cap X(C)|$. Clearly $|X(A)| = |X(B)| = |X(C)| = 3^n$, $|X(A) \cap X(B)| = 2^n = |X(A) \cap X(C)| = |X(C) \cap X(B)|$ and $|X(A) \cap X(B) \cap X(C)| = 1$. Hence, $k = 3^{n+1} + 3 \cdot 2^n + 1$

5. Let X = {1,2, … , 50}, ordered by divisibility. Find the largest length of a chain in X. Find the smallest length of a maximal chain in X
   Solution. Chains are sequences of numbers $(a_1, a_2, \ldots, a_k)$ such that $a_i | a_{i+1}$. Obviously, the smaller the quotient $\frac{a_{i+1}}{a_i}$ the longer our chain. So the best shot is (1,2,4,8,16,32). Another chain of this length is e.g. (1,3,6,12,24,48).
   Hence the answer to the first question is 7. Since every number is divisible by 1, every maximal chain has at least two elements (1 and some other number). On the other hand (1,29) is a maximal chain because 29 is a prime and every multiple of 29 is greater than 50. So the answer to the second question is 2.