EIDMA FINAL EXAM 2, 2015-06-29

1. How many numbers n, $0<n<1000000$ have an odd sum of their digits (in decimal notation)?
   Solution 1. Let us consider numbers n greater than OR EQUAL TO 0, which means we include number 0 in our argument. The set consists of two disjoint parts: O, the set of numbers whose digits add up to an odd number and E – the others. Consider the function f:O→E, defined as follows: if the last digit of n is different from 9 then $f(n) = n+1$, otherwise we just change the last digit of n to 0. Equivalently, $f(n)$ is what we get when we replace the last digit d_l of n with (d_l+1)mod10 and leave the other digits unchanged. For example. $f(8) = 9$, . $f(9) = 0$ (this why we had to include number 0, otherwise $f(n)$ would not be defined for n=9), $f(999999) = 999990$. This function changes (increases or decreases) the number of odd digits in n by 1 and is clearly a bijection. Hence the number of numbers in O is the same as that in E, namely 500000. The task was to find the size of O, the addition of zero affects only E, hence the answer is 500000.
   Solution 2 (less inspiration, more calculation). We consider 6-long sequences of digits instead of numbers. Numbers with fewer digits are left-padded with zeroes, which does not affect the sum of the digits. So the question is reduced to "What is the number of 6-long sequences of digits 0,1, …, 9 containing an odd number of odd digits?" A 6-long sequence must then contain 1 or 3, or 5 odd digits. In how many ways can we construct a sequence with 1 odd digit? Well, we can choose the position for the odd digit in 6 ways and the digit itself in 5 ways (one of 1,3,5,7,9), and in each of the remaining 5 position we place one of five even digits. All told this would be (6*5)*(5*5*5*5*5) = 30*3125 = 93750. Positions for 3 odd digits can be chosen in $\binom{6}{3}$ = 20 ways and the digits themselves in 5*5*5 = 125 ways. In the remaining 3 positions we can put the even digits in 5*5*5 = 125 ways. This gives us 20*125*125 = 312500. The case of 5 odd and one even digit is obviously similar to one odd and 5 even, so the number will be the same, i.e 93750. Hence the total is 312500+93750+93750 = 500000. Surprise, surprise!
2. In how many ways can you distribute 10 identical cookies between Tom, John, Mohammed and James in such a way that Tom gets none, John at least 1, Mohammed at least 2 and James at least 3?
   Solution 1. First we give 1 cookie to John, 2 to Mohammed and 3 to James. Since the cookies are identical this can be done in one way only. This leaves us with 4 identical cookies to be distributed among the three boys (Tom, the naughty one gets nothing). The number of ways to put 4 identical objects into 3 distinguishable containers is given by $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$
   Solution 2. We can first give 1 cookie to Mohammed and 2 to James. Next we have to split the remaining 7 cookies between, John, Mohammed and James so that each gets at least one (Tom, as usual, gets nothing). This is, of course, done by placing the cookies in a row and choosing 2 out of 6 "separators" between them. This approach leads to $\binom{6}{2}$ as well.

3. (Tricky one) Prove or disprove:
   a. for every graph G, if (1,2,2,2,3,4,4,5) is the sequence of degrees of vertices of G then G is connected.
   b. for every graph G, if (1,2,2,2,3,4,4,5) is the sequence of degrees of vertices of G then G is disconnected.

   Solution. First of all there are no graphs with (1,2,2,2,3,4,4,5) as their degree sequence because in every graph the number of vertices of odd degrees must be even and we are dealing with exactly 3 odd numbers here. Hence, paradoxically, both a. and b. are true because the left hand side of the implications is false.

4. Let R and S be equivalence relations on a set X. Prove that $R\cap S$ is an equivalence relation on X. Describe equivalence classes of $R\cap S$ in terms of equivalence classes of R and S.
   Solution. What is the relation $R\cap S$? Relations on X are sets of ordered pairs of elements of X. Hence $(a,b)\in R\cap S$ iff $(a,b)\in R$ and $(a,b)\in S$. In other words, $aR\cap Sb$ iff aRb and aSb. "Is $R\cap S$ an equivalence relation?" means "Is $R\cap S$ reflexive, symmetric and transitive?". Since R and S are equivalence relations they are reflexive, so, for each a, aRa and aSa, i.e. $(a,a)\in R$ and $(a,a)\in S$, hence $(a,a)\in R\cap S$ and $R\cap S$ reflexive. In a similar way one can prove that $R\cap S$ is symmetric. For transitivity, suppose that $aR\cap Sb$ and $bR\cap Sc$. We must prove that $aR\cap Sc$. But $aR\cap Sb$ means, in particular, that aRb, and $bR\cap Sc$ means that bRc. Since R is an equivalence we get aRc. Replacing R with S we can easily get aSc. aRc and aSc means $aR\cap Sc$, i.e. $R\cap S$ is transitive.
   What is the equivalence class of a with respect to $R\cap S$? Well, the definition says $[a]_{R\cap C} = \{x\in X \mid aR\cap Sx\}$ =

$\{x \in X \mid aRx \text{ and } aSx\} = \{x \in X \mid aRx\} \cap \{x \in X \mid aSx\} = [a]_R \cap [a]_S$

5. Find the sets $\bigcap_{t \in R^+} A_t$ and $\bigcup_{t \in R^+} A_t$ where $A_t = \{x \in R : 9\frac{x^2}{t^2} - 1 = 0\}$.

Solution. Obviously, since $t>0$, $A_t = \{x \in R : 9x^2 = t^2\} = \{x \in R : 3|x| = t\}$. For example $A_3 = \{-1,1\}$ and $A_6 = \{-2,2\}$. The intersection is clearly empty and the union is **R**-$\{0\}$.