EIDMA COMMITTEE EXAM 2015-09-22

Solutions

1. Calculate the number of elements of $\{n\in N: 1\le n \ \&\ n\le 1000 \ \&\ \sim(4|n) \ \&\ \sim(6|n) \ \&\ \sim(9|n)\}$

   Solution. Let $A_k = \{n\in N: 1\le n \ \&\ n\le 1000 \ \&\ \sim(k|n)\}$. Obviously A_1 is empty (every integer is divisible by 1, A_2 is the set of odd numbers within {1, … ,1000} and so on. What we are supposed to find is the size of $A_4\cap A_6\cap A_9$. It will be easier to calculate the size of the complement and subtract that from 1000 (=|{1, … , 1000}|). So, we need $|(A_4\cap A_6\cap A_9)'|$ = (by de Morgan's Law) $|A_4' \cup A_6' \cup A_9'|$ = (by inclusion-exclusion principle) $|A_4'|+|A_6'|+|A_9'| - |A_4' \cap A_6'| - |A_4' \cap A_9'| -|A_9' \cap A_6'| + |A_4' \cap A_6' \cap A_9'|$. Now we just need to calculate each term of the above sum. Since A_k is the set on numbers not divisible by k, its complement is the set of numbers divisible by k. So A_4' is the set of numbers (between 1 and 1000) divisible by 4. Clearly they look like p*4, where p=1 ,…, 250, hence $|A_4'| = 250$. Similarly, $|A_6'| = 166$ and $|A_9'| = 111$. Now, what is $A_4'\cap A_6'$? This is the set of numbers divisible by both 4 and 6. Since 4 and 6 have 2 as a common factor, $A_4'\cap A_6'$ consists of all numbers divisible by 12, and $|A_4'\cap A_6'| = \left\lfloor \frac{1000}{12} \right\rfloor = 83$. By the same argument $|A_9'\cap A_6'| = \left\lfloor \frac{1000}{18} \right\rfloor = 55$. 9 and 4 are relatively prime, so $|A_4'\cap A_9'| = \left\lfloor \frac{1000}{36} \right\rfloor = 27$. Finally, a number divisible by 4 and 9 is also divisible by 6, hence $|A_4' \cap A_6' \cap A_9'| = |A_4'\cap A_9'| = 27$. So the final answer is 1000 - (250+166+111-83-55-27+27) = 1000 - (527-112) = 585.

2. Find all numbers k such that the relation R_k defined on the set of natural numbers as: *pR_kq if and only if $k|(p+q)$* is an equivalence relation. For each k for which the answer is YES find the equivalence classes of R_k.

   Solution. R_k must be reflexive, i.e., for every natural number n, k must divide n+n = 2n. Obviously, the only number which divides every even number is 2. Hence our only possible choice for k is k=2. Is R_2 and equivalence relation? It is symmetric because if p+q is even then so is q+p, it is also transitive because p+q and q+r even means that p, q and r are either all odd, or all even. In both cases p+r is even. There are two equivalence classes: the set of even numbers and the set of odd numbers.

3. Find the sets $\bigcap_{t\in R^+} A_t$ and $\bigcup_{t\in R^+} A_t$ where $A_t = \{(x,y)\in R^2 : y - tx \le 0\}$.

   Solution. Clearly, A_t is the half-plane bounded from above by the line passing through the origin, with the slope of t (t>0). No two different lines passing through the origin have a point in common other than the origin itself. Which points belong to all these half-planes? Obviously the origin and every point to the right but not above the origin and also all points below, but not to the left of the origin. Hence $\bigcap_{t\in R^+} A_t = \{(x,y) : x\ge 0 \ \&\ y\le 0\}$.

   Points of the form (0,y), y>0 do not belong to any A_t. Otherwise $y-t0\le 0$ for some t, which would mean $y\le 0$, contrary to y>0. The same goes for points of the form (x,0) for x<0. Points in the second quadrant (x<0, y>0) are also excluded. Other than those, each point of the plane belongs to some A_t (namely for t=y/x if x,y>0 or x,y<0, or for arbitrary positive number t in the case y<0, x>0). Hence the answer is

   $\bigcup_{t\in R^+} A_t = (R^2-\{(x,y):x\le 0 \ \&\ y\ge 0\}) \cup \{(0,0)\}$.