1. For how many numbers n, $0<n<1000000$, the sum of their digits is odd?
   Solution. See Problem 1 in 2015 June 29 exam

2. In how many ways can one paint 15 identical cars blue, yellow, red and green to get at least 1 blue, at least 1 yellow, at least 2 red and at least 3 green cars? (Each car must be painted, in one color).
   Solution. See Problem 2 in 2015 June 29 exam (slightly modified)

3. Vertices of the graph G are binary sequences of length 99, two vertices (sequences) are adjacent iff they differ on exactly one position (e.g. (0,0, … , 0) is adjacent to (1,0,0, … ,0), and to (0,1,0,0, … ,0) etc.).
   a. Is G connected?
   b. Is G Hamiltonian?

   Solution (a) Suppose G is disconnected. Choose vertices *u and v* so that there is no *u-v* path in G and *u and v* differ on the smallest number of positions. Suppose on position *k u* has 0 and *v* has 1. Toggle position *k* in *u*. The resulting sequence *w* is adjacent to *u*, so there is no *w-v* path in G and differs from *v* on fewer positions than *u* – contrary to our choice of *u and v*. Hence G is connected.
   (b) Let G_n be the graph of binary sequences of length *n*. I.e. Our graph G is G_{99}. We will prove by induction on *n* that for each $n \geq 2$ G_n is Hamiltonian, which is more than required. For $n=2$ this is obvious by inspection. Suppose G_n is Hamiltonian for some $n \geq 2$. Consider all vertices of G_{n+1} whose last coordinate is 1. Dropping the last coordinate in each vertex we get a copy of the graph G_n. By induction hypothesis, this subgraph has a Hamiltonian cycle C_1. Suppose $C_1 = (v_0, v_1, \ldots, v_k, v_0)$. We go from v_0 to v_k, then we change the last coordinate of v_k to 0 switching to the corresponding vertex u_0 on the corresponding cycle $C_0 = (u_0, u_1, \ldots, u_k, u_0)$ in the subgraph of G_{n+1} consisting of vertices with last coordinate 0, we go backwards, form u_k to u_0 and switch the last coordinate back to 1, which takes us to v_0.
4. Let *R* and *S* be equivalence relations on a set *X*. Prove that $R \cap S$ is an equivalence relation on *X*. Describe equivalence classes of $R \cap S$ in terms of equivalence classes of *R* and *S*.
   Solution. See Problem 4 in 2015 June 29 exam

5. Prove by induction that any postage of more than 7 cents can be made using only 3 and 5 cent stamps. In other words, every number greater than 7 is a sum of threes and fives.
   Solution. We must prove that every integer $n \geq 8$ can be written as a sum of 3's and 5's. Obviously 8 = 3+5. Suppose n = p3+q5. Consider n+1. If q>0 we write n+1 = (p+2)3 + (q-1)5 (which is n+6-5). If q happens to be equal to 0 p must be at least 3 because $n \geq 8$ and n = 3p. In this case we put n+1 = (p-3)3+2*5.