1. How many ways are there to give 10 different balloons and 15 identical cookies to 8 different children so that each child gets at least 1 cookie?

Solution. Giving balloons to children without any restrictions is like assigning a child to each balloon (not the other way around because some children may not be given balloons, while each balloon must be given to a child). This is of course equivalent to defining a function from the set of 10 balloons to the set of 8 children. The number of such functions is 8^{10}. The second part is different because cookies are identical, so all that counts is the NUMBER of cookies each child gets. There is a ready-made formula for this, but if you prefer thinking to memorizing formulae you line up your 15 cookies and split them into 8 groups by choosing 7 out of 14 gaps between cookies, to be separators. Then, the first group goes to the first child, the second to the second and so on. This guaranties that each child gets a cookie. The number of these choices is of course $\binom{14}{7}$. There is another (but equivalent) approach, which some people tried on the test. First, give a cookie to each child (there is only one way of doing this, since cookies are identical) and then split the remaining 7 cookies into 8 groups (some possibly empty). You can do this by adding extra 7 cookies, forming a 14-long line and then remove arbitrarily chosen 7 as separators, which leads again to $\binom{14}{7}$ choices. Hence the final answer is $8^{10}\binom{14}{7}$.

2. Find the number of binary sequences of length n having exactly k "ones" in m series. (e.g. the series "01100100" has length 8 and has 3 "ones" in 2 series)?

Solution. This is very much like Problem 1. It consists of two subquestions: (a) in how many ways can you form *m* blocks (or series) out of *k* ones, and (b) in how many ways can you separate these blocks of ones with zeroes. The answer to (a) is $\binom{k-1}{m-1}$ (you line-up your k ones and choose *m*-1 out of *k*-1 gaps as separators). To answer (b) you line-up the zeroes, all *n-k* of them, and choose *m* places where you will insert your blocks of ones. Since the sequence may begin or/and end with a block of ones the "places" are not only the *n*-*k*-1 gaps between zeroes but also those before the first, and after the last zero. So the number of choices is $\binom{n-k+1}{m}$. The final answer is the product of the two.

3. Let T be a relation on **R**×**R**-{(0,0)} such that (*x*,*y*)T(*p*,*q*) iff *xq*=*yp*.
   a. Is T an equivalence relation?
   b. If the answer to question 3a. is YES describe equivalence classes of T.

Solution. (a) The answer is YES. Reflexivity and symmetricity are trivial. Transitivity means (a,b)T(p,q) and (p,q)T(x,y) implies ((a,b)T(x,y). In other ways aq=bp and py=qx implies ay=bx. Now, aq=bp and py=qx implies aqpy=bpqx. This means ay=bx if both p and q are different from 0. Suppose p=0. Then 0=b0=aq and 0y = qx = 0. q cannot be equal to 0 because p is, and the pair (0,0) is excluded. Hence a=0. In the same way, since qx=0 and q≠0 we get x=0. So ay=bx=0 also in the case p=0. In the same way we deal with the case q=0.
(b) The equivalence classes are lines passing through (0,0) with the point (0,0) removed.

4. Is it true that for every two propositional functions F and G defined on a set X:
   a. $(\forall x \in X)(F(x) \wedge G(x)) \equiv ((\forall x \in X)(F(x)) \wedge ((\forall y \in X)(G(y))$

   Solution. This is clearly true. The LHS means that $\{x \in X : F(x) \wedge G(x)\} = \{x \in X : F(x)\} \cap \{x \in X : G(x)\} = X$. The intersection of two subsets of X is X iff both are equal to X

   b. $(\forall x \in X)(F(x) \vee G(x)) \equiv ((\forall x \in X)(F(x)) \vee ((\forall y \in X)(G(y))$

   Solution. This is clearly false. Take F(x) = (x>0) , and G(x) = (x<1). Obviously $(\forall x \in R)(x > 0 \vee x < 1)$ is true, while $(\forall x \in R)(x > 0)$ and $(\forall y \in R)(y < 1)$ are both false.

5. A graph G has at least 4 vertices and for every 3 vertices a,b,c the subgraph of G spanned by {a,b,c} has at least 2 edges. Prove that G has a Hamiltonian cycle.

Solution. Let x be any vertex of G and suppose that x is not adjacent to some two vertices y and z. But then the subgraph spanned by $\{x,y,z\}$ has at most one edge, yz, contrary to our assumption. This means that, for every vertex x there is at most one other vertex not adjacent to x. This in turn means that for every vertex x, $\deg(x) \geq n-2$, where n is the number of vertices of G. Since $n \geq 4$ this means that that for every vertex x, $\deg(x) \geq \frac{n}{2}$ and, by Dirac's theorem, G is Hamiltonian.

Solution 2. (Elementary, i.e. without Dirac's theorem). First we prove, as above, that for every vertex x, $\deg(x) \geq n-2$. Let C be a longest simple cycle in G (some simple cycles do exist in G, this follows from considering a subgraph spanned by some 4 vertices). If some vertex t does not belong to C then, by the previous argument, it is adjacent to all but (possibly) one vertices of C, which means there are two consecutive vertices on C adjacent to t, which, in turn, means that C can be extended – contrary to the fact that C was a longest cycle.