EIDMA FINAL EXAM 1 2018-JAN-29 - SOLUTIONS

1. We have unlimited supply of balloons in n colors. Calculate the number of ways to give 2 differently colored balloons to each of k people if
   a. no 2 people can get the same pair of colors
   Solution. We need the number of 1-1 functions from the k-element set of people into the $\binom{n}{2}$-element set of pairs of colors. It is $\binom{n}{2}\left(\binom{n}{2}-1\right)\left(\binom{n}{2}-2\right)\dots\left(\binom{n}{2}-(k-1)\right)$
   b. no 2 people can get the same color.
   Solution 1. Here, if we have already given $2p$ balloons (which also means $2p$ colors) to the first p people, to the $p+1^{st}$ person we can only assign a pair of colors chosen out of the remaining $n-2p$ colors. Hence the total number of choices is $\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}\dots\binom{n-2(k-1)}{2}$.
   Solution 2. Consider a $2k$ long sequence $(c_1,c_2, \dots ,c_{2k})$ of different colors (chosen out of n available ones).There are $n(n-1)(n-2) \dots (n-(2k-1))$ of those. We can then assign the first 2 colors to the first person, the next pair of colors to the second one and so on. Is this our answer? No, because if we change the order of colors c_{2q-1} and c_{2q} for all numbers q from some subset of $\{1,2, \dots ,k\}$ we get the same assignment of colors to people. Which means that every assignment has been counted 2^k times (the number of subsets of $\{1,2, \dots ,k\}$). Hence the answer is $\frac{n(n-1)(n-2)\dots(n-(2k-1))}{2^k}$, which is the same as in solution1.

2. Find the number of permutations of the set $A=\{1, ..., n\}$, where numbers divisible by 5 are not next to each other.
   Solution. To simplify the notation denote the number of numbers not divisible by 5 by p and those divisible by 5 by q. Obviously, $q = \left\lfloor\frac{n}{5}\right\rfloor$ and $p = n-\left\lfloor\frac{n}{5}\right\rfloor$. Line-up all p non-divisibles. This can be done in $p!$ ways. Now we have to squeeze-in the q divisibles. Each divisible will occupy a gap between two non-divisibles or one of the two pseudo-gaps at the beginning and at the end of the line-up. This guarantees that no two numbers divisible by 5 are adjacent. The total number of gaps is $p+1$, hence the number of choices of q gaps is $\binom{p+1}{q}$ and each choice of q gaps can be populated by the q divisibles in $q!$ ways. Hence the answer is $p!q!\binom{p+1}{q}$.

3. Complete the following definitions using ONLY mathematical and logical symbols
   Solutions
   a) $n \bmod k = p$ iff $(\exists q\in\mathbf{Z})(n=kq+p \wedge 0\le p \wedge p<k)$
   b) $\lim_{n\to\infty} a_n = L$ iff $(\forall\varepsilon>0)(\exists m\in\mathbf{N})(\forall n>m)\ |L-a_n|<\varepsilon$
   c) $(X,\ll)$ is a poset. An element p is a *maximal* element of $(X,\ll)$ iff $(\forall x\in X)(\ p\ll x \Rightarrow p=x)$

4. G is a graph whose vertex set is $\{1,2, \dots 100\}$ and whose edge set is $\{\{i,j\}: 0<|i-j|< 5\}$.

   (a) Is G connected?
   Solution. YES. Take any $p,q\in\{1,2, \dots 100\}$ such that $p<q$. Obviously $\{p,p+1\}, \{p+1,p+2\}, \quad , \{p+(q-p)-1,q\}$ are edges of G, hence $(p,p+1, \dots, q-1,q)$ is a p-q path in G.

   b) Is G Eulerian
   Solution. NO. number 2 is adjacent to 1,3,4,5,6 only which means deg(2) = 5 and in an Eulerian graph every vertex has an even degree.

   (c) Is G planar?
   Solution. NO. Vertices 1,2,3,4,5 are all adjacent to each other which means that G contains a K_5. K_5 is known to be nonplanar and every subgraph of a planar graph is planar.

5. $\cong$ is a relation on the set $\mathbf{R}\times\mathbf{R}$ such that $(x,y) \cong (p,q)$ iff $py = xq$. Verify if it is an equivalence relation. If it is describe its equivalence classes.
   Solution. The relation is not transitive. $(1,2) \cong (0,0)$ and $(0,0) \cong (2,1)$ but $\sim[(1,2) \cong (2,1)]$.