Solve the problems on your own, then compare your solutions with mine. TT

Problem 1. (a) Find a Jordan block matrix J similar to $A=\begin{bmatrix} -2 & -1 & -2 & 1 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 3 & -1 \\ 2 & -2 & 5 & -1 \end{bmatrix}$.

(b) Find a matrix P, such that $J = P^{-1}AP$.

Problem 2. (a) Find a Jordan block matrix J similar to $A=\begin{bmatrix} 4 & -1 & -8 & 3 & 2 \\ 3 & 1 & -8 & 2 & 1 \\ 2 & -1 & -4 & 2 & 1 \\ 5 & -2 & -12 & 5 & 1 \\ 1 & -1 & -4 & 2 & 3 \end{bmatrix}$.

(b) Find a matrix P, such that $J = P^{-1}AP$.

Solution.1.
Part (a)
Step 1.

Calculate $\det(A-\lambda I)=\det\begin{bmatrix} -2-\lambda & -1 & -2 & 1 \\ 2 & -\lambda & 3 & -1 \\ 2 & 0 & 3-\lambda & -1 \\ 2 & -2 & 5 & -1-\lambda \end{bmatrix} = \lambda^4$, hence all four eigenvalues are equal to zero.

Step 2.

Calculate the number of Jordan blocks. *rank*(A-0I) = *rank*A = $rank\begin{bmatrix} -2 & -1 & -2 & 1 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 3 & -1 \\ 2 & -2 & 5 & -1 \end{bmatrix} =$

$(r_2+r_1, r_3+r_1, r_4+r_1) = \text{rank}\begin{bmatrix} -2 & -1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix} = (r_3-r_2, r_4-3r_2) = \text{rank}\begin{bmatrix} -2 & -1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2.$

This means that J has two blocks with diagonal entries 0, but sizes of the blocks may be 2×2 and 2×2, or 1×1 and 3×3.

Step 3.

Calculate sizes of Jordan blocks. That requires calculating of ranks of matrices A-0I=A, (A-0I)^2 and so on. It turns out that A^2 is the zero matrix, so its rank is 0. Hence, J has 2 blocks of size at least 2 each that is, J has 2 blocks of size 2 by 2. Hence $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Part (b)

Since we have two 2×2 Jordan blocks, our basis R={v_1,v_2,v_3,v_4} consists of two eigenvectors v_1 and v_3 and their attached vectors v_2 and v_4. The attached vectors are those nonzero solutions of $A^2X=\Theta$, that do not satisfy $AX=\Theta$. We must also take care to choose the attached vectors in such a way that their eigenvectors are linearly independent. Since A^2 is the zero matrix, the first system of equations is trivial (Θ=Θ), so the only condition v_2 and v_4 must satisfy is $AX\neq\Theta$. We can choose v_2=(1,0,0,0) and v_4=(0,1,0,0) getting $v_1 = Av_2$ = (-2,2,2,2) and $v_3 = Av_2$ = (-1,0,0,-2). These vectors form the columns of the change-of-basis matrix P, hence $P = \begin{bmatrix} -2 & 1 & -1 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix}$.

<u>Problem 2</u>. (a) Find a Jordan block matrix J similar to $A = \begin{bmatrix} 4 & -1 & -8 & 3 & 2 \\ 3 & 1 & -8 & 2 & 1 \\ 2 & -1 & -4 & 2 & 1 \\ 5 & -2 & -12 & 5 & 1 \\ 1 & -1 & -4 & 2 & 3 \end{bmatrix}$.

(b) Find a matrix P, such that $J = P^{-1}AP$.

<u>Solution.2.</u>

Part (a)

Step 1.

Calculate *det*(A-λI) = $(1-\lambda)(2-\lambda)^4$. Hence the eigenvalues are λ_1=1 and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 2$.

Step 2.

Since 1 is an eigenvalue of multiplicity 1, there is one 1×1 block for λ=1. We need to know the number of Jordan blocks for λ=2. rank(A-2I) = rank $\begin{bmatrix} 2 & -1 & -8 & 3 & 2 \\ 3 & -1 & -8 & 2 & 1 \\ 2 & -1 & -6 & 2 & 1 \\ 5 & -2 & -12 & 3 & 1 \\ 1 & -1 & -4 & 2 & 1 \end{bmatrix}$ = 3. This means that J has 5-3 = 2 blocks with diagonal entries 2. Their sizes may be 2×2 and 2×2 or 1×1 and 3×3.

Step 3.

Calculate sizes of Jordan blocks. That requires calculating of *rank*(A-2I)-*rank*$(A-2I)^2$ and, possibly *rank*$(A-2I)^2$-*rank*$(A-2I)^3$. It turns out that *rank*$(A-I)^2$ =

$$rank\begin{bmatrix} 2 & -1 & -4 & 1 & 0 \\ -2 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & 8 & -2 & 0 \\ 2 & -1 & -4 & 1 & 0 \end{bmatrix}$$

= 1. Hence, J has 2 blocks of size at least 2 each that is, J has 2 blocks of size 2×2 and there is no need to calculate *rank*$(A-2I)^3$ (If you do, you will see that it is equal to one, and each next power of A-2I has the rank of 1, indicating that there are no blocks of the size 3×3, 4×4 and so on). Hence

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Part (b)

Since we have two 2×2 Jordan blocks and a single 1×1 block, our basis R={v_1,v_2,v_3,v_4,v_5} consists of two eigenvectors v_1 and v_3 for λ=2, their attached vectors v_2 and v_4 and an eigenvector v_5 belonging to the eigenvalue λ=1. The attached vectors are those nonzero solutions of $(A-2I)^2X=\Theta$, that do not satisfy $(A-2I)X=\Theta$. We must also take care to choose the attached vectors in such a way that their eigenvectors are linearly independent.

Since *rank*$(A-2I)^2$ = 1, the system of equations $(A-2I)^2X=\Theta$ is equivalent to 2x-y-4z+1t+0u=0, meaning u=u and, for example y=2x-4z+t, i.e. the general solution is (x,2x-4z+t,z,t,u) = x(1,2,0,0,0)+z(0,-4,1,0,0)+t(0,1,0,1,0)+u(0,0,0,0,1). These vectors are candidates for both eigenvectors and attached vectors, but attached vectors must also satisfy $(A-2I)X\neq\Theta$. We try v_4=(0,0,0,0,1). From $(A-2I)v_4$ we get v_3=(2,1,1,1,1), it is a nonzero vector, so it is a good candidate for v_3. If we try using (0,1,0,1,0) for v_2 we are in for trouble because we get then v_1=(A-2I)(0,1,0,1,0)=(2,1,1,1,1) = v_3 and vectors from a basis must be linearly independent. So we try another vector, v_2=(1,2,0,0,0). We are getting v_1=(0,1,0,1-1), which is linearly independent with v_3. All we need now is an eigenvector for λ=1.

$$A-I=\begin{bmatrix} 3 & -1 & -8 & 3 & 2 \\ 3 & 0 & -8 & 2 & 1 \\ 2 & -1 & -5 & 2 & 1 \\ 5 & -2 & -12 & 4 & 1 \\ 1 & -1 & -4 & 2 & 2 \end{bmatrix}$$

. A-I is row-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

, hence x=u, y=-u, z=0 and t=-2u. Then the general solution looks like u(1,-1,0,-2,1) and we may use (1,-1,0,-2,1) as v_5. Finally, our change-of-basis matrix

$$P=\begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix}$$