1. $R=\{v_1,v_2, \dots v_n\}$ is a basis for V. Prove that $S=\{v_1,v_1+v_2, v_1+v_2+v_3, \dots , v_1+v_2+ \dots +v_n\}$ is also a basis for V.
Solution. Since the size of the set is equal to the size of the basis R it is enough to show that S is linearly independent (or that S spans V). Let $a_1v_1+a_2(v_1+v_2)+ \dots +a_n(v_1+v_2+ \dots +v_n)=\theta$. Easy transformation yields $(a_1+a_2+ \dots +a_n)v_1+(a_2+a_3+ \dots a_n)v_2+ \dots +a_nv_n=\theta$. Since R is linearly independent this implies $a_1+a_2+ \dots +a_n=0$, $a_2+a_3+ \dots a_n=0, \dots ,a_n=0$. Subtracting from each equation the next one we get $a_1=a_2= \dots a_n=0$.
2. $R=\{v_1,v_2, v_3,v_4\}$ is a basis for $\mathbf{R}^4$, T is a linear operator such that $T(v_1)=v_1$, $T(v_2)=v_1+v_2$, $T(v_3)=v_1+v_2+v_3$ and $T(v_4)=v_1+v_2+v_3+v_4$. Find the Jordan block matrix for T.

Solution. From the definition of T follows that $A=M_R[T]=\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$, hence the eigenalues of T are

$\lambda_1=\lambda_2=\lambda_3=\lambda_4=1$. The rank of $A-1I=\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$ is obviously 3, so J has just 4-3=1 Jordan block, i.e.

$J=\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix}$

3. For matrices $J=\begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$ and $A=\begin{bmatrix}-2&-1&-1&1\\1&0&1&-1\\2&2&0&-1\\1&1&1&-2\end{bmatrix}$ find P such that $J=P^{-1}AP$. Verify.
Solution. The form of J implies that all four eigenvalues for J (and thus for A) are equal to –1. The structure of J (1 block 3×3 and 1 block 1×1) means that $R=\{v_1,v_2,v_3,u_1\}$, with v_1 and u_1 eigenvectors and v_2 and v_3 attached vectors of the first and second order resp, and $(A-(-1)I)^3=[0]$. In other words, $(A-(-1)I)^2v_3=v_1$, $(A-(-1)I)v_3 = v_2$ and $(A-(-1)I)u_3=[0]$. Furthermore, v_1 and u_1 must be linearly independent.

$(A-(-1)I) = \begin{bmatrix}-1&-1&-1&1\\1&1&1&-1\\2&2&1&-1\\1&1&1&-1\end{bmatrix}$. $(A-(-1)I)^2 = \begin{bmatrix}-1&-1&0&0\\1&1&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix}$. Let's try $v_3=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, i.e. $v_2=\begin{bmatrix}-1\\1\\2\\1\end{bmatrix}$ and $v_1 = \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$. All we need now is another eigenvector of A, linearly independent with v_1. Row-reduction of

$\begin{bmatrix}-1&-1&-1&1\\1&1&1&-1\\2&2&1&-1\\1&1&1&-1\end{bmatrix}$ easily yields $\begin{bmatrix}1&1&0&0\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}$, which means all eigenvectors (x,y,z,t) satisfy x=-y and z=t.

Our v_1 used y= 1 and t=1, so for u_1 we will take y=0 and t=1, getting $u_1=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$. Hence $P=\begin{bmatrix}-1&-1&1&0\\1&1&0&0\\1&2&0&1\\1&1&0&1\end{bmatrix}$

4. Prove that for every two complex numbers *z* and *v*, if $|z-v|=|z+v|$ then $\frac{z^2}{v^2}$ is a real number.
Solution. Let z=a+bi and v=c+di. $|z-v|=|z+v|$ means that $|z-v|^2=|z+v|^2$, which means $(a-c)^2+(b-d)^2 =$

$(a+c)^2+(b+d)^2$. This resolves to $a^2-2ac+c^2+b^2-2bd+d^2 = a^2+2ac+c^2+b^2+2bd+d^2$ and then $ac+bd=0$. Now $\frac{z^2}{v^2} =$

$$\left(\frac{z}{v}\right)^2 = \left(\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right)^2 = \left(\frac{ac+bd+(bc-ad)i}{c^2+d^2}\right)^2 = \left(\frac{(bc-ad)i}{c^2+d^2}\right)^2 = -\left(\frac{bc-ad}{c^2+d^2}\right)^2 \in \mathbf{R}$$