$$A=\begin{bmatrix}3&2&-1&-4\\-6&-4&6&5\\-4&-2&5&2\\2&1&-1&-2\end{bmatrix},\qquad B=\begin{bmatrix}2&2&1&1\\1&2&1&3\\2&3&1&2\\1&1&1&1\end{bmatrix}\qquad C=\begin{bmatrix}6&4&3&6\\5&5&3&2\\7&6&4&5\\4&2&2&4\end{bmatrix}$$

1. Find a Jordan block matrix J similar to A.

Solution. Eigenvalues are 1,1,1 and –1. The rank of A-1I is equal to 3, -1 is an eigenvalue of multiplicity 1, so

$$J=\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}.$$

2. Find P such that $J=P^{-1}AP$.

Solution. We consider A the matrix of some linear operator F with respect to the standard basis S and J that with respect to another basis $R=\{u_1,u_2,u_3,v_1\}$. Our task is, essentially, to construct the basis R. First we find an eigenvector v_1 belonging to $\lambda=-1$. This is done by solving $(A+I)v_1=\theta$. Next we must find vector u_3 such that $(A+I)^3u_3=\theta$ and $(A+I)^2u_3\neq\theta$. Once we find u_3, we put $u_2=(A+I)u_3$ and $u_1=(A+I)u_2$, and P consists of columns u_1,u_2,u_3 and v_1. There are infinitely many solutions here, but :

- Row reducing A+I one can see that eigenvectors for $\lambda=-1$ look like (t,-2t,0,0), so we may put $v_1=(1,-2,0,0)$.
- $(A-I)^2=\begin{bmatrix}-12&-8&10&12\\4&6&-5&-4\\-8&-4&6&8\\-4&-2&3&4\end{bmatrix}$, $(A-I)^3=\begin{bmatrix}8&8&-8&-8\\-16&-16&16&16\\0&0&0&0\\0&0&0&0\end{bmatrix}$. We can easily see that $u_3=(0,1,1,0)$ is a good choice. Then $u_2=(1,1,2,0)$ and $u_1=(2,1,2,1)$

Hence, one of infinitely many choices for P is $P=\begin{bmatrix}2&1&0&1\\1&1&1&-2\\2&2&1&0\\1&0&0&0\end{bmatrix}$.

3. Solve the matrix equations BX=C. Verify your solution.

Solution. If B is invertible then $X=B^{-1}C$ (NOT $X=CB^{-1}$, matrix multiplication is not commutative!!!)

$$B^{-1}=\begin{bmatrix}3&1&-2&-2\\-2&-1&2&1\\-2&-1&1&3\\1&1&-1&-1\end{bmatrix},\ B^{-1}C=X=\begin{bmatrix}1&1&0&2\\1&1&1&0\\2&-1&1&3\\0&1&0&-1\end{bmatrix}$$

4. $F:V\to W$ and $G:W\to T$ are linear mappings. Prove that $G\circ F$ is a linear mapping and $\ker(F)\subseteq\ker(G\circ F)$.

Solution. $G\circ F(v+u)=G(F(v+u))=G(F(v)+F(u))=G(F(v))+G(F(u))=G\circ F(v)+G\circ F(u)$. Similarly $G\circ F(av)=G(F(av))=G(aF(v))=aG(F(v))=a(G\circ F(v))$, hence $G\circ F$ is linear.

If $v\in \ker F$ then $F(v)=\theta$ and $G(F(v))=G(\theta)=\theta$, hence $v\in\ker(G\circ F)$.

5. Solve (in complex numbers) the equation $(\bar{z})^4|z|^2=z^2$.

Solution. Taking modulus of both sides yields $|z|^6=|z|^2$, which implies |z|=0 , i.e. z=0 ,or |z|=1. Plugging |z|=1 into our original equation results in $(\bar{z})^4=z^2$. Multiplying both sides by z^4 we get $(\bar{z})^4z^4=z^6$, i.e. $z^6=1$. You conclude by listing all six roots of 1 of order 6.