1. Number *i* is a root of the equation $z^4-2z^3+3z^2-2z+2=0$. Find the remaining roots.
Solution. Since the coefficients are all real, $-i$ is also a root, so the polynomial is divisible by $(z+i)(z-i)=z^2+1$. The division yields the polynomial z^2-2z+2, whose roots (by standard method) are $1+i$ and $1-i$.

2. Find the Jordan block matrix for $A=\begin{bmatrix}1 & -1 & -1 & 1\\ 1 & 3 & 1 & -1\\ 2 & 2 & 3 & -1\\ 1 & 1 & 1 & 1\end{bmatrix}$.

Solution. The characteristic polynomial turns is $(2-\lambda)^4$, so all four eigenvalues are equal to 2. The rank(A-2I)=2, $(A-2I)^2=\begin{bmatrix}-1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\end{bmatrix}$, its rank is equal to 1. This means that we have 4-2=2 blocks, out of which 2-1=1 has size at least 2, so J must be equal to $\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}$.

3. $A=\begin{bmatrix}-1 & 1 & -1 & 0\\ -2 & -5 & 1 & 1\\ -2 & -4 & 0 & 1\\ -6 & -10 & 1 & 2\end{bmatrix}$ is similar to $B=\begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & -1\end{bmatrix}$. Find a matrix P, such that $B=P^{-1}AP$.
Solution. Since B is a Jordan block matrix similar to A, eigenvalues of A are the same as those of B, i.e. all are equal to –1. Since the square of B+I is the zero matrix then so is the square of $A+I=\begin{bmatrix}0 & 1 & -1 & 0\\ -2 & -4 & 1 & 1\\ -2 & -4 & 1 & 1\\ -6 & -10 & 1 & 3\end{bmatrix}$, which means the attached vectors v_2 and v_4 can be chosen at random, as long as the resulting eigenvectors $v_1=(A+I)v_2$ and $v_3=(A+I)v_4$ are linearly independent. A good choice for v_2 and v_4 is, for example, $v_2=(0,0,1,0)$ and $v_4=(0,0,0,1)$, which yields $v_1=(-1,1,1,1)$ and $v_3=(0,1,1,3)$. Hence a possible $P=\begin{bmatrix}-1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 0 & 3 & 1\end{bmatrix}$.

4. T is a linear mapping such that for every linearly independent set S, T(S) is also linearly independent. Prove that T is one-to one.
Solution. If T is not one-to-one, there exist different vectors *u* and *v* such that T(*u*)=T(*v*). Hence $T(u-v)=\theta$, which contradicts our assumption, because $\{u-v\}$ is linearly independent and $\{\theta\}$ is not.

5. F(1,0,0,0)=(1,1,1,0), F(0,1,0,0)=(1,1,0,1), F(0,0,1,0)=(1,0,1,1) and F(0,0,0,1)=(0,1,1,1), F is a linear operator. Find the matrix of F in the basis R={(1,1,1,0),(1,1,0,1),(1,0,1,1),(0,1,1,1)}.
Solution. Denote $v_1=(1,1,1,0)$, $v_2=(1,1,0,1)$, $v_3=(1,0,1,1)$ and $v_4=(0,1,1,1)$. To find the matrix in question we must calculate $T(v_1)$, express it as a linear combination of $v_1,\dots,v_4$ and put the coefficients in the first column. Then do the same for v_2,v_3 and v_4. For example, $T(v_1)=T(1,1,1,0)=T((1,0,0,0)+(0,1,0,0)+(0,0,1,0)) = T(1,0,0,0) + T(0,1,0,0) + T(0,0,1,0) = (1,1,1,0)+(1,1,0,1)+(1,0,1,1)=v_1+v_2+v_3+0v_4$. In the same way we obtain $T(v_2)=v_1+v_2+0v_3+v_4$, $T(v_3) = v_1+0v_2+v_3+v_4$ and $T(v_4)=0v_1+v_2+v_3+v_4$. Hence $M_R(T)=\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\end{bmatrix}$.