1. Let X be a nonempty set and let **F** denote the set of all functions from X into **R**. For every f,g∈F we define (f+g)(x)=f(x)+g(x), and (f∗g)(x)=f(x)g(x). Under what conditions is (**F**,+,∗) a field?
   Solution. The condition is that X is a one element set. If it is then all functions from X into **R** are constant and resulting algebra is isomorphic with **R**. Otherwise X contains at least two different elements *a* and *b*. The function f(*a*)=0 and f(*x*)=1 for every $x \neq a$ is not the zero function and is not invertible with respect to ∗.
2. Sets A and B are linearly independent. Prove that if *span*(A)∩*span*(B)={θ} then A∪B is linearly independent.
   Solution. Let A={v_1, … v_k} and B={w_1, … ,w_p}. Suppose $a_1v_1+ .. +a_kv_k+b_1w_1+ … +b_pw_p=\theta$. Then $a_1v_1+ .. +a_kv_k=(-b_1)w_1+ … +(-b_p)w_p$. Since LHS is a vector from span(A) and RHS is from *span*(B), they both belong to *span*(A)∩*span*(B). Since the only vector there is θ, both are equal to θ, and, by linear independence of A and B we get $a_1= … =a_k=b_1= … =b_p=0$, i.e. A∪B is linearly independent.
3. F(x,y,z,t)=(-2x-3y+4z-4t,x+z-t,x+y-z+2t,x+2y-3z+4t). Find the Jordan block matrix for F. *Hint* $\lambda^3(\lambda-1)=0$.
   Solution. Obviously, $A=M_S(F)=\begin{bmatrix}-2 & -3 & 4 & -4\\ 1 & 0 & 1 & -1\\ 1 & 1 & -1 & 2\\ 1 & 2 & -3 & 4\end{bmatrix}$. One easily verifies that rank(A-0I)=3 and rank(A-1I)=3. Hence, the number of J-blocks for λ=0 is 1 and the number of J-blocks for λ=1 is also 1. That means that $J=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$.
4. For the matrices $J=\begin{bmatrix}-2 & 1 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & -2 & 1\\ 0 & 0 & 0 & -2\end{bmatrix}$ and $A=\begin{bmatrix}-3 & -5 & 1 & 4\\ -2 & -6 & 2 & 4\\ 1 & -1 & -3 & 0\\ -3 & -6 & 3 & 4\end{bmatrix}$ find P such that $J=P^{-1}AP$.
   Solution. $(A-(-2)I)^2$ is the zero matrix (because so is $(J-(-2)I)^2$). Hence P has columns $(A-(-2)I)v_2, v_2, (A-(-2)I)v_4, v_4$, where v_2 and v_4 are chosen so that $(A-(-2)I)v_2$ and $(A-(-2)I)v_4$ are linearly independent. We may choose $v_2=(1,0,0,0)$ and $v_4=(0,0,0,1)$, getting $P=\begin{bmatrix}-1 & 1 & 4 & 0\\ -2 & 0 & 4 & 0\\ 1 & 0 & 0 & 0\\ -3 & 0 & 6 & 1\end{bmatrix}$.
5. $M_R(T)=\begin{bmatrix}1 & 2 & 4 & 3\\ 2 & 3 & 3 & 4\\ 3 & 4 & 2 & 1\\ 4 & 5 & 1 & 2\end{bmatrix}$, R={$v_1,v_2,v_3,v_4$). Find $M_S(T)$, where S={v_2,v_3,v_4,v_1}.
   Solution. The first column of $M_R(T)$ signifies that $T(v_1)=v_1+2v_2+3v_3+4v_4 = 2v_2+3v_3+4v_4+1v_1$. But v_1 is the last vector in S, so 2,3,4,1 will form the last column in $M_S(T)$. Doing the same for v_2,v_3 and v_4 we get $M_S(T)=\begin{bmatrix}3 & 3 & 4 & 2\\ 4 & 2 & 1 & 3\\ 5 & 1 & 2 & 4\\ 2 & 4 & 3 & 1\end{bmatrix}$.