HINTS & SOLUTIONS

1. *Problems of the type* : A given complex number is a root of a polynomial with real coefficients, find remaining roots.
   Hint: If the coefficients of the polynomial are real (and ONLY in this case) we can use the fact that the conjugate of a root is also a root.
   Example. 1-i is a root of $z^4-6z^3+16z^2-20z+12$. Find the remaining roots.
   SOLUTION. First, the degree of our polynomial is 4, so, by the main theorem of algebra we need three more roots. Since all coefficients are real and 1-i is a root, so one of the remaining three roots is 1+i . Hence, our polynomial is divisible by $(z-(1-i))(z-(1+i)) = (z-1+i)(z-1-i) = z^2-2z+2$. The division yields z^2-4z+6. For this quadratic polynomial, $\Delta=-8$, $\sqrt{\Delta} = \pm 2\sqrt{2}i$, so the missing two roots are $2-\sqrt{2}\,i$ and $2+\sqrt{2}\,i$.
2. *Problems of the type* : Show that the intersection of two substructures (like groups, fields, vector spaces) is also a substructure.
   Hint. These problems require the WTH (What-The-Hell) approach. You need to realize what-the-Hell is the intersection of two sets (it consists of those objects who simultaneously belong to both sets), and also what-the-Hell is this particular type of structure. In the case of subspaces of a vector space you may use the theorem we proved in class.
   Example. V is a vector space over a field F. Show that for every two subspaces W and U of V, $W\cap U$ is a subspace.
   SOLUTION. $\Theta\in W$ and $\Theta\in U$ so $\Theta\in W\cap U$ and $W\cap U \neq \emptyset$. If $x,y\in W\cap U$ then $x,y\in W$ and $x,y\in U$. Then $x+y\in W$ because W is a subspace and $x+y\in U$ because U is a subspace. Hence $x+y\in W\cap U$, i.e. $W\cap U$ is closed under vector addition. In the same way one shows that $W\cap U$ is closed under scaling.

3. *Problems of the type* : Calculate some power of a quotient of two complex numbers.
   Hint. Usually it helps to find polar forms of the two complex numbers, use the division lemma to find the polar form of the quotient and then use de Moivre Law to calculate the solution. Sometimes (Example 2 - Problem 1 from the year 2005) it is easier to divide first, represent the quotient in the polar form and then use de Moivre Law.
   Example 1.
   Calculate $\left(\dfrac{\sqrt{2}+i\sqrt{2}}{-1+i\sqrt{3}}\right)^{1272}$

   SOLUTION. $\left(\dfrac{\sqrt{2}+i\sqrt{2}}{-1+i\sqrt{3}}\right)^{1272} = \left(\dfrac{\frac{\sqrt{2}+i\sqrt{2}}{2}}{\frac{-1+i\sqrt{3}}{2}}\right)^{1272} = \left(\dfrac{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}}{\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}}\right)^{1272} = \left(\cos(\frac{\pi}{4}-\frac{2\pi}{3})+i\sin(\frac{\pi}{4}-\frac{2\pi}{3})\right)^{1272} =$

   $$= \left(\cos(-\frac{5\pi}{12})+i\sin(-\frac{5\pi}{12})\right)^{1272} = \cos(-\frac{1272\cdot 5\pi}{12})+i\sin(-\frac{1272\cdot 5\pi}{12}) = \cos(-530\pi)+i\sin(-530\pi) = 1$$

   Example 2.
   Calculate $\left(\dfrac{1+\sqrt{3}+(1-\sqrt{3})i}{2+2i}\right)^{59}$

   SOLUTION. $\left(\dfrac{1+\sqrt{3}+(1-\sqrt{3})i}{2+2i}\right)^{59} = \left(\dfrac{1+\sqrt{3}+(1-\sqrt{3})i}{2+2i}(2-2i)\right)^{59} = \left(\dfrac{(1+\sqrt{3}+(1-\sqrt{3})i)(2-2i)}{8}\right)^{59} =$

   $$= \left(\frac{2+2\sqrt{3}+2i-2\sqrt{3}i-2i-2\sqrt{3}i+2-2\sqrt{3}}{8}\right)^{59} = \left(\frac{4-4\sqrt{3}i}{8}\right)^{59} = \left(\frac{1-\sqrt{3}i}{2}\right)^{59} = \left(\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3})\right)^{59} =$$

   $$= \cos\left(-\frac{59\pi}{3}\right)+i\sin\left(-\frac{59\pi}{3}\right) = \cos\left(\frac{(-60+1)\pi}{3}\right)+i\sin\left(\frac{(-60+1)\pi}{3}\right) = \cos\left(-20\pi+\frac{\pi}{3}\right)+i\sin\left(-20\pi+\frac{\pi}{3}\right) =$$

   $= \cos\frac{\pi}{3}+i\sin\frac{\pi}{3} = \frac{1}{2}+\frac{\sqrt{3}}{2}i$. Easy as π.
4. *Problems of the type* : Find all complex numbers z satisfying an equation involving z, $\bar{z}$, $|z|$, some exponents and coefficients.
   Hint. These equations can usually be reduced to $z^n=c$ for some constants n and c. Use identities $z\bar{z}=|z|^2$, $|z|=|\bar{z}|$, $|zw|=|z||w|$ and some common sense. Once you reduce the equation to the form $z^n=c$ use the root formula to find all n

roots of c.

Example. Problem 2(2009) $i(\bar{z})^3 z = 8|z|$

SOLUTION. Apply modulus to both sides. We get $|i(\bar{z})^3 z| = |8|z||$, which yields $|z|^4=8|z|$. This means $|z|=0$ or $|z|^3=8$, i.e. $|z|=2$. In the case $|z|=0$ we get $z=0$ – and this is one of our solutions. Now consider the case $|z|=2$. Plugging this into our original equation we get

$i(\bar{z})^3 z = 16$

$i(\bar{z})^2 \bar{z} z = 16$

Since $z\bar{z}=|z|^2 = 4$ we get

$i(\bar{z})^2 4 = 16$ and

$i(\bar{z})^2=4$ and

$(\bar{z})^2 = -4i$

Conjugating both sides we get

$z^2 = 4i$. Roots of $4i$ are $z_1=\sqrt{2}+i\sqrt{2}$ and $z_2=-\sqrt{2}-i\sqrt{2}$. So our solutions are: 0, $\sqrt{2}+i\sqrt{2}$ and $-\sqrt{2}-i\sqrt{2}$.