# LAG – Midterm 1. NOV 18, 2019 – SOLUTIONS

1) Solve (in complex numbers) the equation $(x+1+i)^4+16 = 0$.
*Solution.* We put $t= x+1+i$. The equation becomes $t^4=-16$. The polar form for $-16$ is $16(\cos\pi + i\sin\pi)$ which means the arguments modulus of each of the four roots is 2 and the arguments are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Hence $t_0=\sqrt{2} + i\sqrt{2}$, $t_0=-\sqrt{2} + i\sqrt{2}$, $t_0=-\sqrt{2} - i\sqrt{2}$, $t_0=\sqrt{2} - i\sqrt{2}$ and finally $x_k=t_k-(1+i)$ for each $k=0,1,2,3$.

2) Is $(\{a+bi\in\mathbf{C}: (ab=0 \vee a^2=b^2) \wedge a^2+b^2 \neq 0\}, * )$ a group? (* stands for multiplication of complex numbers).
*Solution.* Denote $X = \{a+bi\in\mathbf{C}: (ab=0 \vee a^2=b^2) \wedge a^2+b^2 \neq 0\}$. The main problem here is to verify that X is closed under multiplication, i.e., if $a+bi$ and $c+di$ belong to X then so does $(a+bi)(c+di)$.
**Method 1**. The easiest way to do this is to notice that $ab=0$ means either $a=0$ or $b=0$ (but not both, $a^2+b^2 \neq 0$), so $\mathrm{Arg}(a+bi)$ is $0, \frac{\pi}{2}, \pi$ or $3\frac{\pi}{2}$, while $a^2=b^2$ means that $\mathrm{Arg}(a+bi)$ is $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$. Combining these we get that elements of X are nonzero complex numbers with arguments in the form $k\frac{\pi}{4}$ for some integer $k$. From the multiplication lemma (the argument of the product is the sum of arguments of the factors) we get that the argument of $(a+bi)(c+di)$ is something like $k\frac{\pi}{4} + l\frac{\pi}{4} = (k+l)\frac{\pi}{4}$ and, of course, the product of two nonzero complex numbers is itself different from 0.
**Method 1**. Take $a+bi, c+di \in X$. Then ($a=0$ OR $b=0$ OR $a=b$ OR $a=-b$) AND ($c=0$ OR $d=0$ OR $c=d$ OR $c=-d$) which means we must consider all 16 possibilities: ($a=0$ AND $c=0$), ($a=0$ AND $d=0$), … , ($a=-b$ AND $c=-d$). In each case we must prove that $(a+bi)(c+di) = (ac-bd) + (ad+bc)i \in X$, i.e.
(1) $(ac-bd) = 0$ OR
(2) $(ad+bc) = 0$ OR
(3).$(ac-bd) = (ad+bc)$ OR
(4) $(ac-bd) = -(ad+bc)$.
For example consider case ($b=0$ AND $c=-d$), we must show that at least one of (1), (2), (3), (4) is satisfied by plugging $b=0$ and $c=-d$ into (1), (2), (3), (4). By trial and error we notice that in (4) LHS = $a(-d)-0d$ and RHS = $-(ad+0c)$, both of which are equal to $-ad$ so (4) holds true. For each of the remaining 15 cases you have to find which (if any) of the 4 conditions (1) - (4) is satisfied. Each individual case it is easy but there is an awful lot of them. You may imagine now why I do not recommend this method.
The rest is easy. Associativity is obvious. The identity element $1 = 1+0i$ belongs to X because $1*0=0$, and the inverse for $r(\cos k\frac{\pi}{4} + i\sin k\frac{\pi}{4})$ is $\frac{1}{r}(\cos(-k\frac{\pi}{4}) + i\sin(-k\frac{\pi}{4}))$ which also belongs to X.

3) Solve (in complex numbers) the equation $z^4-iz^2+2 = 0$.
*Solution (outline)*. Putting $t = z^2$ we get $t^2-it+2 = 0$. Solving this the standard way one gets $t_0 = -i$ and $t_1 = 2i$. Then we just calculate roots of order 2 of these two numbers by de Moivre.

4) Let (G,#) be a group. Let X denote the set of all bijections (1-1 and "onto" functions) $f : G \to G$ such that for every $a,b\in G$ we have $f(a\#b) = f(a)\#f(b)$. Show that $(X,\circ)$ is a group ($\circ$ denotes composition of functions).
*Solution.* First, is X closed under composition? Composition of two bijections is a bijection, it was verified on the lecture on permutations.
Our bijections from X must satisfy the additional condition $f(a\#b) = f(a)\#f(b)$. Consider $f,g\in X$ and check if $f\circ g\in X$, i.e. if $f\circ g\in X$. $(f\circ g)(a\#b) = f(g(a\#b)) = f(g(a)\#g(b)) = f(g(a))\#f(g(b)) = (f\circ g)(a)\#(f\circ g)(b)$ as required.
Associativity is obvious (function composition is always associative).
The identity element is the identity function $id(x) = x$. Obviously id is a bijection and $id(a\#b) = a\#b = id(a)\# id(b)$.
The inverse for a bijection f is of course the inverse function $f^{-1}$ which also is a bijection. But does it belong to X, i.e. does it satisfy the condition that for every $a$ and $b$ from G $f^{-1}(a\#b) = f^{-1}(a)\#f^{-1}(b)$? Well, suppose it doesn't. Then there exist some $x,y\in G$ such that $f^{-1}(x\#y) \neq f^{-1}(x)\#f^{-1}(y)$. Since f is a bijection it is 1-1 so applying f to two different arguments, $f^{-1}(x\#y)$ and $f^{-1}(x)\#f^{-1}(y)$ should yield two different values. But $f(f^{-1}(x\#y)) = x\#y$ and $f(f^{-1}(x)\#f^{-1}(y)) = x\#y$ so both values are equal to $x\#y$ which contradicts injectivity of f.