Midterm test 2 (E&IT) 12-JAN-2011

1. W and U are subspaces of V, $W\cap U=\{\Theta\}$, $\{w_1,\dots,w_k\}$ is a basis for W, $\{u_1,\dots,u_m\}$ is a basis for U. Prove that the set $\{w_1,\dots,w_k\}\cup\{u_1,\dots,u_m\}$ is linearly independent.
   Solution.
   What the ... are we supposed to prove? That the set $\{w_1,\dots,w_k,u_1,\dots,u_m\}$ is linearly independent.
   What the ... does that mean? That for all possible $a_1,\dots a_k,b_1,\dots,b_m$ from our field of scalars **F** $a_1w_1+\dots+a_kw_k+b_1u_1+\dots+b_mu_m=\Theta$ implies $a_1=\dots=a_k=b_1=\dots=b_m=0$.
   Start thinking: Suppose $a_1w_1+\dots+a_kw_k+b_1u_1+\dots+b_mu_m=\Theta$. This means $a_1w_1+\dots+a_kw_k=-b_1u_1-\dots-b_ku_m$. Since the vector on the left hand side belongs to W, and the one on the right hand side belongs to U they both belong to $W\cap U$, which consists of Θ only. Hence $a_1w_1+\dots+a_kw_k=\Theta$ and $b_1u_1+\dots+b_mu_m=\Theta$. Since both $\{w_1,\dots,w_k\}$ and $\{u_1,\dots,u_m\}$are linearly independent we get $a_1=\dots=a_k=0$ and $b_1=\dots=b_m=0$.

2. Solve the equation $\det\begin{bmatrix}1-x & 1 & 1 & 1\\ 1 & 1-x & 1 & 1\\ 1 & 1 & 1-x & 1\\ 1 & 1 & 1 & 1-x\end{bmatrix}=0$.
   Solution. Careful calculation of the determinant yields $x^4-4x^3=0$, which means x=0 or x=4.

3. $A=\begin{bmatrix}\alpha & 2\\ 2 & 4\end{bmatrix}, B=\begin{bmatrix}\alpha & 1\\ 2 & 1\end{bmatrix}$. Find all real numbers α such that the matrix equation AX=B has exactly one solution.

   Solution. Denote $X=\begin{bmatrix}x & z\\ y & t\end{bmatrix}$. Then our matrix equation is equivalent to the system of equations $\begin{cases}\alpha x+2y=\alpha\\ 2x+4y=2\\ \alpha z+2t=1\\ 2z+4t=1\end{cases}$.

   It is uniquely solvable iff $0\neq\begin{vmatrix}\alpha & 2 & 0 & 0\\ 2 & 4 & 0 & 0\\ 0 & 0 & \alpha & 2\\ 0 & 0 & 2 & 4\end{vmatrix}$ =($r_1-\frac{\alpha}{2}r_2$, $r_3-\frac{1}{2}r_4$) $=\begin{vmatrix}0 & 2-2\alpha & 0 & 0\\ 2 & 4 & 0 & 0\\ 0 & 0 & \alpha-1 & 0\\ 0 & 0 & 2 & 4\end{vmatrix}$ =(2α-2) $\begin{vmatrix}2 & 0 & 0\\ 0 & \alpha-1 & 0\\ 0 & 2 & 4\end{vmatrix}$ = $8(\alpha-1)^2$. Hence the answer is **R**-{1}.

4. Find dim(span{1,2,1,2),(2,1,2,1),(1,3,1,3),(1,1,1,1)}).

   Solution. We just calculate the rank of the matrix $\begin{bmatrix}1 & 2 & 1 & 2\\ 2 & 1 & 2 & 1\\ 1 & 3 & 1 & 3\\ 1 & 1 & 1 & 1\end{bmatrix}$, which turns out to be 2.