1) Show that for every vector space V and for every set of vectors $\{v_1, v_2, \dots, v_n\}$ $\mathrm{span}(v_1, v_2, \dots, v_n) = \mathrm{span}(v_1, v_1+v_2, v_2+v_3, \dots, v_{n-1}+v_n)$.

Solution. Let $A = \mathrm{span}(v_1, v_2, \dots, v_n)$ and $B = \mathrm{span}(v_1, v_1+v_2, v_2+v_3, \dots, v_{n-1}+v_n)$. We must show that $B \subseteq A$ and $B \supseteq A$. If $w \in B$ then $w = a_1v_1 + a_2(v_1+v_2) + a_3(v_2+v_3) + \dots + a_n(v_{n-1}+v_n) = (a_1+a_2)v_1 + (a_2+a_3)v_3 + \dots + a_nv_n \in B$. In the other direction, if $u \in A$ then $u = b_1v_1 + b_2v_2 + \dots + b_nv_n$. We have to express u as a linear combination of $v_1, v_1+v_2, v_2+v_3, \dots, v_{n-1}+v_n$, hence we must find $c_1, \dots c_n$ such that $b_1v_1 + b_2v_2 + \dots + b_nv_n = c_1v_1 + c_2(v_1+v_2) + c_3(v_2+v_3) + \dots + c_n(v_{n-1}+v_n) = (c_1+c_2)v_1 + (c_2+c_3)v_3 + \dots + c_nv_n$. One possibility is to find find $c_1, \dots c_n$ such that $c_1+c_2 = b_1$, $c_2+c_3 = b_2$, , $c_{n-1}+c_n = b_{n-1}$, $c_n = b_n$. Clearly, putting $c_n = b_n$, $c_{n-1} = b_{n-1} - c_n = b_{n-1} - b_n$, …, $c_1 = b_1 - b_2$ does the trick.

2) Find A^{-1} for $A = \begin{bmatrix} 4 & 3 & 3 & 3 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$. Verify your solution by matrix multiplication.

Solution. It is easy to notice that rows of A are linearly dependent ($r_1 = r_2 + r_5$) so the matrix is not invertible.

3) W and U are subspaces of V, $W \cap U = \{\Theta\}$, $\{w_1, \dots, w_k\}$ is a basis for W, $\{u_1, \dots, u_m\}$ is a basis for U. Prove that the set $\{w_1, \dots, w_k, u_1, \dots, u_m\}$ is a basis for $W+U = \{w+u : w \in W \wedge u \in U\}$.

Solution. Obviously $\mathrm{span}(\{w_1, \dots, w_k, u_1, \dots, u_m\}) = W+U$. We must show that $\{w_1, \dots, w_k, u_1, \dots, u_m\}$ is linearly independent. Suppose, for some $a_1, \dots, a_k, b_1, \dots b_m$, $a_1w_1 + \dots a_kw_k + b_1u_1 + \dots + b_mu_m =$. Then $a_1w_1 + \dots a_kw_k = -b_1u_1 - \dots - b_mu_m$. But this means that $a_1w_1 + \dots a_kw_k$ and $-b_1u_1 - \dots - b_mu_m$ both belong to $W \cap U$, hence both are equal to Θ (the zero vector) and, consequently, $a_1, \dots, a_k, b_1, \dots b_m = 0$

4) Solve (in **C**) the equation $\det\begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & x & 1 \\ 1 & x & 1 & 1 \\ x & 1 & 1 & 1 \end{bmatrix} = 0$.

Solution. Row operations r_1-r_2, r_2-r_3 and r_3-r_4 yield $\begin{vmatrix} 0 & 0 & 1-x & x-1 \\ 0 & 1-x & x-1 & 0 \\ 1-x & x-1 & 0 & 0 \\ x & 1 & 1 & 1 \end{vmatrix}$. Taking out the common factor in rows 1,2,and 3 we get $(1-x)^3 \begin{vmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ x & 1 & 1 & 1 \end{vmatrix}$ r_1+r_4 yields

$(1-x)^3 \begin{vmatrix} x & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ x & 1 & 1 & 1 \end{vmatrix} = (1-x)^3 \begin{vmatrix} x & 1 & 2 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = (1-x)^3(-x-3)$. The roots are 1 and -3.