Chapter 1
Complex Numbers – continued

Here comes one of the most important ideas in abstract algebra

Definition 1.1. Fields $(F,+,\cdot)$ and $(X,\#,*)$ are said to be *isomorphic* iff there exists a bijection (i.e. a one-to-one and "onto" function) $f:F\to X$ such that

$(\forall a,b\in F)\ f(a+b)=f(a)\#f(b)$ and $f(a\cdot b)=f(a)*f(b)$

Every such function is then called an *isomorphism*.

The definition of an isomorphism can be easily applied to groups or other algebraic systems. Algebras that are isomorphic are considered "essentially identical". They differ only in secondary respects, such as the nature of the elements, the labels we use to denote them, the symbols we use for operations and such like, while in "what counts" they are identical. From algebra point of view "what counts" is properties of the operations, not only those listed in the definition of the particular type of algebra but all of them.

Example 1.1. Consider groups $(\mathbf{R},+)$ and $(\mathbf{R}^+,\cdot)$. The function $f(x)=2^x$ is a bijection and $f(a+b)=2^{a+b}=2^a\cdot 2^b=f(a)\cdot f(b)$, hence f is an isomorphism. You can think of $(\mathbf{R},+)$ as an exact model of $(\mathbf{R}^+,\cdot)$. That means, you can predict the result of multiplication of two positive numbers watching the result of addition of their representatives in **R**. In other words, you can live without ability to multiply, as long as you can add and you don't mind calculating powers and logarithms. Suppose you want to multiply 0.5 by 8. First, we must find out who represents 0.5 and 8. Since $f(-1)=2^{-1}=0.5$ and $f(3)=2^3=8$, 0.5 and 8 are represented by –1 and 3, respectively. Now, $0.5\cdot 8=f(-1)f(3)=f(-1+3)=f(2)=2^2=4$.

Example 1.2. Consider $(\mathbf{Z}_2,\oplus,\otimes)$ and $(\{a,b\},\#,*)$, where # and * are defined as follows

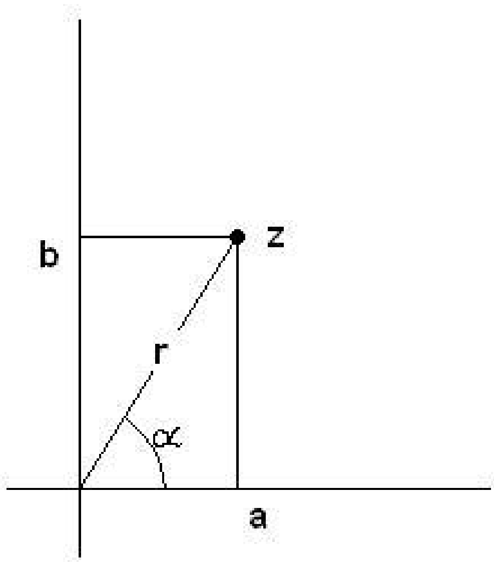
#	a	b
a	b	a
b	a	b

*	a	b
a	a	b
b	b	b

Since $(\{a,b\},\#,*)$ is isomorphic to $(\mathbf{Z}_2,\oplus,\otimes)$ (the isomorphism being $f(a)=1$, $f(b)=0$) we can claim that $(\{a,b\},\#,*)$ is a field, because all algebraic properties are preserved by an isomorphism.

Example 1.3. $(R\times R,+,\cdot)$ where + and $\cdot$ are defined "componentwise", i.e. $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)\cdot(c,d) = (a\cdot c, b\cdot d)$ is NOT a field,since no element of the form (0,b) or (a,0) is invertible.

Example 1.4. (R×R,+,·) with componentwise addition and multiplication defined as follows: (a,b)·(c,d) = (ac-bd,ad+bc) is a field. This field is isomorphic to the field of complex numbers, the isomorphism being f(a+b*i*)=(a,b).



We can look at the field from last example as another approach to complex numbers. We identify complex numbers with points of the Cartesian plane (or vectors anchored at the origin) and we call this "geometrical interpretation of complex numbers". A point *z* of the plane can be identified by its Cartesian coordinates, say (a,b), but also by its polar coordinates, i.e. the distance *r* from the origin and the angle α between positive half-axis OX and the segment (0,0)(a,b). Hence, (a,b)=(r*cos*α,r*sin*α) or, equivalently, z = a+b*i* = r(*cos*α + i*cos*α). The last expression is known as the polar form of the complex number z. The nonnegative number r is called the absolute value or modulus of z, and is denoted by |z|. Clearly if z is given in the standard form z=a+b*i* then |z|= $\sqrt{a^2+b^2}$. The angle α is called an argument of z. Since both sine and cosine are periodic function with the period of 2π, a complex number has infinitely many arguments. The argument of z that belongs to the interval <0;2π) is called the principal argument of z.

With every complex number z=a+b*i* we associate its conjugate number $\bar{z}$ = a-b*i*. Geometrically $\bar{z}$ is the mirror image of z with respect to the X axis.

Theorem 1.1 The function f(z) = $\bar{z}$ is an isomorphism of **C** with itself.

Proof. It is enough to verify by hand that $\overline{z+w}=\bar{z}+\bar{w}$ and $\overline{zw}=\bar{z}\,\bar{w}$.□

Fact $z\bar{z}=|z|^2$.

Proof. $(a+bi)(a-bi) = a^2+b^2$ □

Example 1.5. Here are polar forms of some complex numbers:

$1=\cos 0+i\sin 0$

$-1=\cos\pi+i\sin\pi$

$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$1+i=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

If $z=r(\cos\alpha+i\sin\alpha)$ then $\bar{z} = r(\cos\alpha-i\sin\alpha) = r(\cos(-\alpha)+i\sin(-\alpha))$

Theorem 1.2 (de Moivre Law)

For every positive integer n if $z=r(\cos\alpha+i\sin\alpha)$ then $z^n=r^n(\cos n\alpha+i\sin n\alpha)$.

The theorem follows easily from the following lemma.

Lemma 1.1. For every two complex numbers $z=r(\cos\alpha+i\sin\alpha)$ and $w=p(\cos\beta+i\sin\beta)$ we have $zw=rp(\cos(\alpha+\beta)+i\sin(\alpha+\beta))$.

Proof of the lemma.

$zw=r(\cos\alpha+i\sin\alpha)p(\cos\beta+i\sin\beta) = rp((\cos\alpha\cos\beta-\sin\alpha\sin\beta)+i(\cos\alpha\sin\beta+\sin\alpha\cos\beta)) =$

$rp(\cos(\alpha+\beta)+i\sin(\alpha+\beta))$. The last transformation follows from well-known trigonometric identities.□

De Moivre Law can be used also to calculate roots of complex numbers.

Definition 1.2. Every complex number w satisfying the equation $w^n=z$ is called a root of z of order n.

Suppose $z=r(\cos\alpha+i\sin\alpha)$ and $w=p(\cos\beta+i\sin\beta)$ is a root of z of order n. Then $w^n = p^n(\cos n\beta + i\sin n\beta) = r(\cos\alpha+i\sin\alpha)$. Hence $p=\sqrt[n]{r}$ (in the usual sense) and $\cos n\beta=\cos\alpha$ and $\sin n\beta=\sin\alpha$.

Since 2π is the period of sin and cos, we get $n\beta_k=\alpha+k2\pi$, or $\beta_k=\frac{\alpha+k2\pi}{n}$, for k=0,1,2, ...

Notice that for every integer p, $\beta_{k+pn}=\frac{\alpha+(k+pn)2\pi}{n}=\frac{\alpha+k2\pi+pn2\pi}{n}=\frac{\alpha+k2\pi}{n}+p2\pi$.

Hence, $w_k = \sqrt[n]{r}(\cos\beta_k+i\sin\beta_k) = \sqrt[n]{r}(\cos\beta_{k+pn}+i\sin\beta_{k+pn}) = w_{k+pn}$. This indicates that we only get n different roots of z of order n, namely $w_0, w_1, \dots, w_{n-1}$ – no more, no less.

Example 1.6. Find $\sqrt[4]{-1}$.

First -1=cosπ+$i$sinπ. Hence $\beta_k = \dfrac{\pi + k2\pi}{4}$ for k=0,1,2,3. We get four solutions

$z_0 = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$,

$z_1 = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4} = -\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$,

$z_2 = \cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$

$z_3 = \cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4} = \dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$