Chapter 1
Linear Mappings

<u>Definition 1.1.</u> Let V and W be vector spaces over a field **F**. A function $\varphi:V\to W$ is called a <u>linear mapping</u> iff

(a) $(\forall u,v\in V)\ \varphi(u+v)=\varphi(u)+\varphi(v)$, and

(b) $(\forall v\in V)(\forall p\in \mathbf{F})\ \varphi(pv)=p\varphi(v)$

<u>Proposition 1.1.</u> A function $\varphi:V\to W$ is a linear mapping iff

(c) $(\forall p,q\in \mathbf{F})\ (\forall u,v\in V)\ \ \varphi(pu+qv) = p\varphi(u)+q\varphi(v)$

<u>Proof.</u> ($\Rightarrow$) Suppose φ is a linear mapping. Then $\varphi(pu+qv) = \varphi(pu)+\varphi(qv) = p\varphi(u) + q\varphi(v)$, by (a) and (b) in that order.

($\Leftarrow$) To prove (a) we simply put p=q=1 in (c) and to prove (b) we put q=0 and u=v. Then $\varphi(pv) = =\varphi(pv+\Theta) = \varphi(pv+0v) = p\varphi(v)+0\varphi(v) = p\varphi(v)+\Theta = p\varphi(v)$.□

<u>Example 1.1.</u> $V=W=R_n[x]$, $\varphi(f(x))=f\,'(x)$. Differentiation is obviously a linear mapping

<u>Example 1.2.</u> $\varphi:\mathbf{R}^3\to\mathbf{R}^4$, $\varphi(x,y,z)=(x+y,2x-z,x+y+z,y)$. We have $\varphi(x,y,z)+(a,b,c)) = \varphi(x+a,y+b,z+c) = ((x+a)+(y+b), 2(x+a)-(z+c), (x+a)+(y+b)+(z+c), (y+b)) = (x+y+a+b, 2x-z+2a-c, x+y+z+a+b+c, y+b) = \varphi(x,y,z) + \varphi(a,b,c)$ and $\varphi(p(x,y,z))= \varphi(px,py,pz) = (px+py,2px-pz,px+py+pz,py) = =(p(x+y),p(2x-z),p(x+y+z),py) = p\varphi(x,y,z)$. Hence φ is a linear mapping.

<u>Example 1.3.</u> $\varphi:\mathbf{R}^3\to\mathbf{R}^2$, $\varphi(x,y,z)=(x+y-1,x-z)$. φ is not a linear mapping as $\varphi(\Theta+\Theta) = \varphi(\Theta) = \varphi(0,0,0)= (0+0-1,0-0) = (-1,0)$, while $\varphi(\Theta)+\varphi(\Theta)=(-1,0)+(-1,0) = (-2,0)$. This is enough to show that φ is not linear, but let us note that φ does not satisfy the second condition, too, as $\varphi(2\Theta)=\varphi(\Theta)=(-1,0)$ and $2\varphi(\Theta)=2(-1,0)=(-2,0)$.

<u>Example 1.4.</u> $V=2^{\{1,2,\ldots,n\}}$, the space of all subsets of $\{1,2,\ldots,n\}$, over the field $\mathbf{Z}_2$ (see Example 2 in Chapter 4), $W= \mathbf{Z}_2$, $\varphi(A)=|A|\text{mod}2$. There are two cases to be considered while verifying the second condition, namely p=0 or p=1. In the case p=0 we have $\varphi(0A)= \varphi(\varnothing)=0$ and $0\varphi(A)=0(|A|\text{mod}2)=0$. In the second case $\varphi(1A)= \varphi(A) = |A|\text{mod}2 = 1(|A|\text{mod}2)=1\varphi(A)$, so the second condition holds. To verify the first condition let us consider $\varphi(A\div B) = |(A\div B)|\text{mod}2 = |(A\cup B)-(A\cap B)|\text{mod}2$. Since $A\cap B$ is a subset of $A\cup B$ we have $|(A\cup B)-(A\cap B|=|A\cup B|-|A\cap B|$.

From the inclusion-exclusion principle we have $|A\cup B| = |A|+|B|-|A\cap B|$. Hence $\varphi(A\div B) = (|A|+|B|-2|A\cap B|)\mathrm{mod}2 = (|A|+|B|)\mathrm{mod}2 = (|A|\mathrm{mod}2 + |B|\mathrm{mod}2)\mathrm{mod}2 = \varphi(A)\oplus\varphi(B)$.

We will now define two important parameters of a linear mapping, *rank* and *nullity*.

Definition 1.2. Let $\varphi:V\to W$ be a linear mapping. The image of φ is the set $\mathrm{im}\varphi = \varphi(V)$ and the kernel of φ is the set $\ker\varphi=\{v\in V: \varphi(v)=\Theta_W\}$

Proposition 1.2. $\mathrm{im}\varphi$ is a subspace of W and $\ker\varphi$ is a subspace of V.

Proof. Let $w_1,w_2\in \mathrm{im}\varphi$. There exist $v_1,v_2\in V$ such that $\varphi(v_i)=w_i$, i=1,2. Then $pw_1+qw_2 = p\varphi(v_1)+ q\varphi(v_2) = \varphi(pv_1+qv_2)\in \mathrm{im}\varphi$, hence $\mathrm{im}\varphi$ is a subspace of W. For every v_1 and v_2 from $\ker\varphi$ we have $\varphi(av_1+bv_2) = a\varphi(v_1)+b\varphi(v_2) = a\Theta+b\Theta = \Theta$, so $\ker\varphi$ is a subspace of V.□

Definition 1.3. $\mathrm{rank}(\varphi)=\dim(\mathrm{im}\varphi)$, $\mathrm{nullity}(\varphi)=\dim(\ker\varphi)$

Example 1.5. Let $V=W=\mathbf{R}_n[x]$ and $\varphi(f(x))=f'(x)$. Then $\ker\varphi=\mathbf{R}_0[x]$, the subspace consisting of all constant polynomials and $\mathrm{im}\varphi=\mathbf{R}_{n-1}[x]$. Obviously, $\mathrm{rank}(\varphi)=n$ and $\mathrm{nullity}(\varphi)=1$.

Proposition 1.3. For every set $\{v_1,v_2, \dots ,v_n\}\subseteq V$, and for every linear mapping $\varphi:V\to W$, $\varphi(\mathrm{span}(v_1,v_2, \dots ,v_n))=\mathrm{span}(\varphi(v_1),\varphi(v_2), \dots , \varphi(v_2))$.

Proof. $w\in \varphi(\mathrm{span}(v_1,v_2, \dots ,v_n))$ iff there exist scalars $a_1,a_2, \dots a_n$ such that $w=\varphi(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i\varphi(v_i)$. That means $w\in \varphi(\mathrm{span}(v_1,v_2, \dots ,v_n))$ iff $w\in \mathrm{span}(\varphi(v_1),\varphi(v_2), \dots , \varphi(v_2))$.□

Example 1.6. Let $\varphi:\mathbf{R}^3\to\mathbf{R}^3$, $\varphi(X) = AX$, where $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 1 & 3 & 5\end{bmatrix}$ *and* $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$. In the traditional notation, $\varphi(x,y,z)=(x+2y+3z,2x+3y+4z,x+3y+5z)$. To find the kernel of φ we must solve the matrix equation $AX=\mathbf{0}$, in other words we must solve the system of linear equation

$\begin{cases} x+2y+3z=0\\ 2x+3y+4z=0\\ x+3y+5z=0\end{cases}$ This system is equivalent to $\begin{cases} x+2y+3z=0\\ -y-2z=0\\ y+2z=0\end{cases}$ and then to $\begin{cases} x \quad - z = 0\\ y + 2z = 0\end{cases}$

which means x=z, y=–2z and z ranges over **R**. Hence every vector from kerφ has the form (z,–2z,z) = z(1,–2,1) i.e. kerφ=span((1,–2,1)) and nullity(φ)=1.

To find imφ we notice that, by Proposition 1.3 imφ=span(φ(1,0,0), φ(0,1,0), φ(0,0,1)) = span((1,2,1),(2,3,3),(3,4,5)). If the three spanning vectors were linearly independent then rankφ would be equal to 3, but (1,2,1)=2(2,3,3)–(3,4,5) hence, by Theorem 4.5, ??? imφ = span((2,3,3), (3,4,5)). Since the last two vectors are obviously linearly independent, rank(φ)=2.

<u>Theorem 1.1.</u> Let S={v_1,v_2, … ,v_n} be a basis for V and let f be a function, mapping S into W, i.e. f:{v_1,v_2,…,v_n}→W. Then there exists exactly one linear mapping φ:V→W such that for each i=1,2, … ,n φ(v_i)=f(v_i)

<u>Proof.</u> We must prove two things, that there exists such linear mapping φ and that it is unique. To do the first,. let us take any vector v from V. Since S is a basis for V, $v=\sum_{i=1}^{n} a_i v_i$ for some scalars a_1,a_2, … a_n. We define $\varphi(v)=\sum_{i=1}^{n} a_i f(v_i)$. By Theorem 4.6<?>, the scalars a_1,a_2, … a_n are uniquely determined by v, hence φ is a function. Is φ a linear mapping? Let us take another vector u∈V. There exist scalars b_1,b_2, … ,b_n such that $u=\sum_{i=1}^{n} b_i v_i$. Let us take any two scalars p,q∈**F** and consider

φ(pu+qv) = $\varphi(p(\sum_{i=1}^{n} b_i v_i)+q(\sum_{i=1}^{n} a_i v_i))$ =

= $\varphi(\sum_{i=1}^{n}(pb_i+qa_i)v_i)$ = by properties of vector addition and scaling

= $\sum_{i=1}^{n}(pb_i+qa_i)f(v_i)$ = by the definition of φ

= $\sum_{i=1}^{n}(pb_i f(v_i)+qa_i f(v_i))$ = by properties of vector addition and scaling

= $p\sum_{i=1}^{n} b_i f(v_i)+q\sum_{i=1}^{n} a_i f(v_i)$ = by properties of vector addition and scaling

= pφ(u)+ qφ(v) by the definition of φ. Hence, by Proposition 1.1, φ is a linear mapping. To prove the uniqueness of φ let us suppose that there is another linear mapping ψ, such that $\psi(v_i)=f(v_i)$ for i=1,2, … ,n. Consider arbitrarily chosen vector $v=\sum_{i=1}^{n} a_i v_i$. By linearity of ψ,

$\psi(v)= \sum_{i=1}^{n} a_i\psi(v_i)=\sum_{i=1}^{n} a_i f(v_i) = \varphi(v)$.□

Theorem 1.1 says that every linear mapping from V into W is uniquely determined by its values on vectors from a basis of V, or, equivalently, that every function defined on a basis is uniquely extendable to a linear mapping.

Theorem 1.2. rank(φ)+nullity(φ)=dimV

Proof. Let $\{v_1,v_2, \dots ,v_n\}$ be a basis for kerφ. Then there exist vectors $w_1,w_2, \dots ,w_k$ such that $\{v_1,v_2, \dots ,v_n,w_1,w_2, \dots ,w_k\}$ is a basis for V. It is enough to show that $\{\varphi(w_1), \varphi(w_1), \dots , \varphi(w_k)\}$ is a basis for imφ. By Proposition 1.3 it is enough to show that $\{\varphi(w_1), \varphi(w_1), \dots , \varphi(w_k)\}$ is linearly independent. Let $\sum_{i=1}^{k} a_i\varphi(w_i)=\Theta$. By linearity of φ we have $\Theta = \sum_{i=1}^{k} a_i\varphi(w_i) = \varphi(\sum_{i=1}^{k} a_i w_i)$, i.e. $\sum_{i=1}^{k} a_i w_i \in \ker\varphi$. Hence, for some scalars $b_1,b_2, \dots ,b_n$ we have $\sum_{i=1}^{k} a_i w_i = \sum_{j=1}^{n} b_j v_j$.

This implies that $\sum_{i=1}^{k} a_i w_i + \sum_{j=1}^{n}(-b_j)v_j = \Theta$, hence all coefficients a_i and b_j are equal to 0, in particular all a_i-s are zeroes.□

Theorem 1.3. A linear mapping φ:V→W is one-to-one iff φ preserves linear independence, i.e. for every linearly independent subset S of V, φ(S) is a linearly independent subset of W.

Proof. (⇒) Suppose $S=\{v_1,v_2, \dots ,v_n\}$ is a linearly independent subset of V and $\sum_{i=1}^{n} a_i\varphi(v_i) =\Theta_W$.

We must prove that $a_1= \dots =a_n=0$. Since φ is a linear mapping we have $\sum_{i=1}^{n} a_i\varphi(v_i) = \varphi(\sum_{i=1}^{n} a_i v_i)$ $= \Theta$. Since $\varphi(\Theta_V)=\Theta_W$ and φ is a 1-1 function we have $\sum_{i=1}^{n} a_i v_i = \Theta_V$ and, by linear independence of S, we have $a_1= \dots =a_n=0$.

(⇐) Suppose S={v_1,v_2, … ,v_n} is a basis of V and for some v and u from V, $v=\sum_{i=1}^{n} a_i v_i$ and $u=\sum_{i=1}^{n} b_i v_i$, and φ(v)=φ(u). Then Θ_W = φ(v)–φ(u) = φ(v–u) = $\sum_{i=1}^{n}(a_i - b_i)\varphi(v_i)$. Since φ preserves linear independence we obtain that a_i=b_i for i=1,2, …,n and that means that u=v.□

Theorem 1.4. A linear mapping φ:V→W is one-to-one iff kerφ={∅}

Proof. (⇒) From Theorem 1.3 and Proposition 1.3 it follows that for every basis {v_1,v_2, … ,v_n} of V {φ(v_1),φ(v_2), … ,φ(v_n)} is a basis for imφ, hence dim(kerφ)=n–dim(imφ)=0 and that means that kerφ={∅}.

(⇐) If kerφ≠{∅} then there exists v≠**0** such that φ(v)=**0**. Since φ(**0**)=**0,** φ is not one-to-one.□