SYSTEMS OF LINEAR EQUATIONS

Problem 1. Prove that matrix multiplication is associative.
Problem 2. A square matrix A is said to be *symmetric* iff $A=A^T$. Is it true that multiplication of symmetric matrices is commutative?
Problem 3. Show that $(AB)^T=B^TA^T$.
Problem 4. Reduce the matrices to their row echelon and row canonical forms and find their ranks

$$A=\begin{bmatrix}2&1&3&-2&1\\1&1&2&3&-3\\2&2&2&2&2\\5&7&-3&-6&4\end{bmatrix}\quad B=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{bmatrix}\quad C=\begin{bmatrix}5&1&2&6&1&0\\4&0&1&2&1&1\\3&5&2&3&2&1\\2&4&1&-1&2&2\end{bmatrix}$$

$$D=\begin{bmatrix}1&3&-2&5&4\\1&4&1&3&5\\1&4&2&4&3\\2&7&-3&6&13\end{bmatrix}\quad E=\begin{bmatrix}1&4&5&-1\\1&5&8&-2\\2&5&1&2\end{bmatrix}\quad F=\begin{bmatrix}2&1\\3&-7\\-6&1\\5&-8\end{bmatrix}$$

Problem 5. Find a basis and the dimension to the solution space of the systems of linear equations:

$$(a)\ \begin{cases}x+2y-2z+2s-t=0\\x+2y-z+3s-2t=0\\2x+4y-7z+s+t=0\end{cases}\qquad (b)\ \begin{cases}x+2y-z+3s-4t=0\\2x+4y-2z-s+5t=0\\2x+4y-2z+4s-2t=0\end{cases}$$

Problem 6. Find a homogeneous system of linear equations whose solution space is spanned by the set of vectors: {(1,-2,0,3,-1),(2,-3,2,5,-3),(1,-2,1,2,-2)}.
Problem 7. Show that for every two n×k matrices A and B,

$$\operatorname{rank}(A+B)\le\operatorname{rank}(A)+\operatorname{rank}(B).$$

Problem 8. Let A be a square n×n matrix. Show that if $AX=\Theta$ has only the zero solution, then, for every vector B, $AX=B$ has a unique solution.
Problem 9. Discuss solvability of the following systems of equations in terms of parameters *a* and *b*.

$$(a)\ \begin{cases}ax+by+az+t=1\\ax+y+az+t=1\\x+y+az+at=1\end{cases}\qquad (b)\ \begin{cases}x+2y+3z+4t=a\\2x+3y+4z+5t=b\\3x+4y+5z+6t=a\\4x+5y+6z+7t=b\end{cases}\qquad (c)\ \begin{cases}x+ay+az=1\\ax+ay+z=1\\ax+y+az=1\\ax+ay+az=1\end{cases}$$

Problem 10. Find general solutions to the following systems of equations:

$$(a)\ \begin{cases}2x+2y+3z+4t=2\\2x+3y+4z+6t=0\\3x+4y+5z+6t=1\\x+y+z+t=1\end{cases}\qquad (b)\ \begin{cases}x+y+z-t+u=1\\x-y+z+t+u=1\\-x+y-z+t-u=0\end{cases}\qquad (c)\begin{cases}x+y+z+t+u=0\\x-2y+3z-4t+5u=0\end{cases}$$