ALGEBRAS AND GROUPS

Problem 1. In each of the following determine whether $*$ is a binary operation on the indicated set X. If it is then indicate whether $*$ is associative and whether it is commutative. Also discuss the existence of identity and inverses.

(a) $m * n = mn+1$, X=$\mathbf{Z}$
(b) $m * n = n$, X=$\mathbf{Z}$
(c) $m * n = m+n-1$, X=$\mathbf{Z}$
(d) $m * n = 2m+n$, X=$\mathbf{Z}$
(e) $m * n = (m+n)/2$, X=$\mathbf{Z}$
(f) $m * n = 2^{mn}$, X=$\mathbf{Z}$
(g) $(x,y) * (p,q) = (xp,yq)$, X=$\mathbf{Z}\times\mathbf{Z}$
(h) $(x,y) * (p,q) = (xp,y+q)$, X=$\mathbf{Z}\times\mathbf{Z}$
(i) $(x,y) * (p,q) = (xq,y+p)$, X=$\mathbf{R}\times\mathbf{R}$
(j) $(x,y) * (p,q) = (x+p,y+q)$, X=$\mathbf{R}\times\mathbf{Q}$
(k) $(x,y) * (p,q) = (x+q,y+p)$, X=$\mathbf{R}\times\mathbf{R}$
(l) $(x,y) * (p,q) = (x+p,y+q)$, X=$\mathbf{R}\times\mathbf{R}$
(m) $x * y=x+y+xy$, X=$\mathbf{R}$

Problem 2. Show that if $a * a = e$ for all a from a group G, then G is abelian.

Problem 3. A function f mapping the XY plane onto itself is called an isometry iff f preserves distances between points, i.e. $dist(P_1,P_2) = dist(f(P_1),f(P_2))$.Show that all isometries of the plane form a group (with function composition as the group operation).

Problem 4. In those algebras from problem 1 which are groups, find some nontrivial subgroups.

Problem 5. Find all subgroups in the following groups
(a) $(\mathbf{Z}_6, \oplus)$
(b) $(\mathbf{Z}_7, \oplus)$
(c) $(\mathbf{Z}_6^{\#}, \otimes)$
(d) $(\mathbf{Z},+)$
(e) $(S_3, \circ)$

Problem 6. Determine which of the following pairs of groups are isomorphic
(a) $(\mathbf{Z}_6, \oplus)$ and $(\mathbf{Z}_7^{\#}, \otimes)$
(b) $(\mathbf{Z}_6, \oplus)$ and $(\mathbf{Z}_{12}, \oplus)$
(c) $(\mathbf{Z}_6, \oplus)$ and $(\mathbf{Z},+)$
(d) $\mathbf{Z}\times\mathbf{R}$ and $\mathbf{R}\times\mathbf{Z}$ with operations defined as componentwise addition (like in part (l) of problem 1).
(e) $(\mathbf{Z},+)$ and $(\mathbf{Z}, *)$, where $x * y=x+y-1$ (first verify that $(\mathbf{Z}, *)$ is a group).
(f) $(\mathbf{Z}_8, \oplus)$ and $(2^X, \div)$, where X={1,2,3}.

Problem 7. Let H and K be two subgroups of a group G
(a) show that $H\cap K$ is a subgroup of G
(b) show that $H\cup K$ is not necessarily a subgroup of G.