POLYNOMIALS AND VECTOR SPACES

Problem 1. Let *f(x)*$\in$**R**[x]. Show that, if the degree of *f* is odd then *f* has at least one root in **R**.
Problem 2. Calculate
(a) $(x+1)^2$ in $\mathbf{Z}_2[x]$
(b) $(x-1)^3$ in $\mathbf{Z}_3[x]$. Remember that in $\mathbf{Z}_3$ (-1)=2, so x-1=x+2.
(c) $(1+x+x^2+x^3)(2-x+3x^2)$ in $\mathbf{Z}_2[x]$.
Problem 3. Perform the long division
(a) $x^5+2x^4+3x^3+2x^2+x+1$ by $2x^2+3x+1$
(b) x^4+1 by $2x+3$
in $\mathbf{Z}_3[x]$, then in $\mathbf{Z}_5[x]$ and then in **C**[x] ("3" is defined as 1+1+1, "2" is defined as 1+1).
Problem 4. Find all roots of
(a) x^4+1
(b) x^2+3x+1
(c) $x^5+2x^4+3x^3+2x^2+x+1$
in $\mathbf{Z}_3[x]$, in $\mathbf{Z}_5[x]$.
Problem 5.
Problem 6. Prove that for every prime number *n* there exists a polynomial *f(x)*$\in$$\mathbf{Z}_n$[x] such that for each *k*$\in$ {0,1, ... ,n-1} *f(k)*$\neq$0.
Problem 7. Determine whether or not the following sets are vector spaces over indicated fields:
(a) **Z** over $\mathbf{Z}_2$, with ordinary addition, and scalar multiplication defined as follows: 0r=0, 1r=r for every r$\in$**Z**,
(b) $\mathbf{Z}_4$ over $\mathbf{Z}_2$ with natural operations,
(c) **C** over **R**,
(d) **R** over **C**,
(e) $\mathbf{Q}_\mathbf{b}$[x] (all polynomials from **Q**[x] with degree $\leq$n) over **R**,
(f) $\mathbf{Q}_\mathbf{n}$[x] over **Q**,
(g) All polynomials from **R**[x] with exactly two different roots, over **R**,
(h) $\mathbf{R}_\mathbf{b}$[x] (all polynomials from **R**[x] with degree $\leq$n) over **R**,
(i) All functions f:$\mathbf{R}\to\mathbf{R}$ such that f(0)=1, over **R**,
(j) All functions f:$\mathbf{R}\to\mathbf{R}$ such that f(1)=0, over **R**,
(k) All continuous functions f:$\mathbf{R}\to\mathbf{R}$, over **R**,
(l) All functions f:$\mathbf{R}\to\mathbf{R}$ which are discontinuous at 0, over **R**,
(m) All subsets of a set X, over $\mathbf{Z}_2$, with the symmetric difference as vector addition, and with 0A=$\emptyset$ and 1A=A for every A$\subseteq$X,
(n) All real sequences (a_n) such that $\lim_{n\to\infty} a_n$ =0, over **R**,

Problem 8. Prove that in a vector space p(v-u)=pv-pu.
Problem 9. The intersection of any collection of subspaces of some linear space V is a subspace of V. Show that this is not necessarily true for the union of subspaces,
Problem 10. The real plane, $\mathbf{R}^2$, is a vector space over **R**. Describe, in geometrical terms, all subspaces of $\mathbf{R}^2$.