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Multivariate reciprocal inverse Gaussian distributions from the Sabot–Tarrès–Zeng integral

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ABSTRACT

In Sabot and Tarrès (2015), the authors have explicitly computed the integral

$$STZ_n = \int \exp(-\langle x, y\rangle)(\det M_x)^{-1/2} dx$$

where M_x is a symmetric matrix of order n with fixed non-positive off-diagonal coefficients and with diagonal $(2x_1, \ldots, 2x_n)$. The domain of integration is the part of $\mathbb{R}^n$ for which M_x is positive definite. We calculate more generally for $b_1 \geq 0, \ldots b_n \geq 0$ the integral

$$\int \exp\left(-\langle x, y\rangle - \frac{1}{2} b^\top M_x^{-1} b\right)(\det M_x)^{-1/2} dx,$$

we show that it leads to a natural family of distributions in $\mathbb{R}^n$, called the $MRIG_n$ probability laws. This family is stable by marginalization and by conditioning, and it has number of properties which are multivariate versions of familiar properties of univariate reciprocal inverse Gaussian distribution. In general, if the power of $\det M_x$ under the integral in STZ_n is distinct from $-1/2$ it is not known how to compute the integral. However, introducing the graph G having $V = \{1, \ldots, n\}$ for set of vertices and the set E of $\{i, j\}'$ s of non-zero entries of M_x as set of edges, we show also that in the particular case where G is a tree, the integral

$$\int \exp(-\langle x, y\rangle)(\det M_x)^{q-1} dx$$

where $q > 0$, is computable in terms of the MacDonald function K_q.

1. Introduction: the Sabot–Tarrès–Zeng integral

Let us describe the integral appearing in Sabot and Tarrès [14]. Let $W = (w_{ij})_{1\leq i,j\leq n}$ be a symmetric matrix such that $w_{ii} = 0$ for all $i = 1, \ldots, n$ and such that $w_{ij} \geq 0$ for $i \neq j$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define the matrix $M_x = 2\,\mathrm{diag}(x_1, \ldots, x_n) - W$. For instance if $n = 3$ we have

$$M_x = \begin{bmatrix} 2x_1 & -w_{12} & -w_{13} \\ -w_{12} & 2x_2 & -w_{23} \\ -w_{13} & -w_{23} & 2x_3 \end{bmatrix}.$$

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Denote by C_W the set of $x \in \mathbb{R}^n$ such that M_x is positive definite. It is easy to see that C_W is an open non-empty unbounded convex set. This is not a cone in general. Frequently we consider the undirected graph G with set of vertices $\{1, \dots, n\}$ and with set of edges $E = \{\{i, j\} \; ; \; w_{ij} > 0\}$ and we speak of the graph G associated to W. The Sabot–Tarrès–Zeng integral is, for $y_1, \dots, y_n > 0$

$$STZ_n = \int_{C_W} e^{-(x_1 y_1 + \cdots + x_n y_n)} \frac{dx_1 \times \cdots \times dx_n}{\sqrt{\det M_x}} = \left(\sqrt{\frac{\pi}{2}}\right)^n \frac{1}{\sqrt{y_1 \times \cdots \times y_n}} e^{-\frac{1}{2}\sum_{ij} w_{ij}\sqrt{y_i y_j}}. \tag{1}$$

Sabot and Tarrès [14] give a probabilistic proof of this remarkable result. Another proof is in Sabot, Tarrès and Zeng [15], based on the Cholesky decomposition. This integral leads naturally to consideration of probability laws on $\mathbb{R}^n$ that we call STZ_n distributions with densities proportional to $e^{-\langle x,y \rangle}(\det M_x)^{-1/2} 1_{C_W}(x)$. In the present paper we derive, using a different approach than the two methods mentioned above, a more general $MRIG_n$ integral in Theorem 2.2. In particular, we give a new proof of (1). The symbol $MRIG$ for multivariate reciprocal inverse Gaussian, is explained below.

This $MRIG_n$ integral enables us to create a new set (called the $MRIG_n$ family) of distributions on $\mathbb{R}^n$ which is stable by marginalization and, up to a translation, stable by conditioning. The bibliography concerning the appearance of the STZ_n and $MRIG_n$ laws in probability theory is already very rich and we suggest to look at Sabot and Zeng [16] and Disertori, Merkl and Rolles [7] for many references. An unpublished observation of 2015 of the first author has been used and reproved in these two publications and some facts of the present paper can be found in them. However, we use here only elementary methods to get our results.

Let us recall that in literature, the generalized inverse Gaussian distributions $GIG(a, b, q)$ are one dimensional laws with density proportional to $e^{-a^2 x - \frac{b^2}{4x}} x^{q-1} 1_{(0,\infty)}(x)$, for $a, b > 0$ and q real (see Seshadri [18] for instance). Parameterizations differ according to the needs of authors and we have chosen an appropriate one in the present paper. The most famous particular case is for $q = -1/2$ with the inverse Gaussian distribution. A random variable Y with the inverse Gaussian distribution $IG(a, b) = GIG(a/2, 2b, -1/2)$ has Laplace transform

$$\mathbb{E}(e^{-sY}) = e^{b(a - \sqrt{a^2 + s})} \tag{2}$$

for $s > -a^2$. Its density is proportional to $e^{-\frac{a^2 y}{4} - \frac{b^2}{y}} y^{-3/2} 1_{(0,\infty)}(y)$. A less known case, but the important one for the present paper, is for $q = 1/2$ with the reciprocal inverse Gaussian distribution. Actually, it is a distribution of the inverse of a random variable with an IG distribution. A random variable X with a reciprocal inverse Gaussian distribution $RIG(a, b) = GIG(a, b, 1/2)$ has Laplace transform for $s > -a^2$

$$\mathbb{E}(e^{-sX}) = \frac{a}{\sqrt{a^2 + s}} e^{b(a - \sqrt{a^2 + s})} \tag{3}$$

and is such that

$$\mathbb{E}(X) = m = \frac{ab + 1}{2a^2}, \; \mathbb{E}(X^2) = \frac{1}{4a^4}(a^2 b^2 + 3ab + 3), \; \mathbb{V}\mathrm{ar}(X) = \frac{ab + 2}{4a^4}. \tag{4}$$

Its density is proportional to $e^{-a^2 x - \frac{b^2}{4x}} x^{-1/2} 1_{(0,\infty)}(x)$. This law is considered for instance in Barndorff–Nielsen and Koudou [3]. Our $MRIG_n$ distributions have some properties which are multivariate versions of properties known for the univariate RIG law. These are good reasons for attaching the name multivariate (n-dimensional) RIG to the members of this family. A particular case of the family $MRIG_2$ appears in Barndorff–Nielsen, Blaesild and Seshadri [2]. The family STZ_2 appears in Barndorff–Nielsen and Rysberg [4].

Section 2 proves and comments on the $MRIG_n$ integral, including a presentation of the Disertori–Spencer–Zinbauer [8] and Disertori–Merkles–Rolles [7] integrals in the studies of supersymmetry. Section 3 gives some examples. Section 4 details the properties of the $MRIG_n$ laws (we carefully distinguish along the paper the $MRIG_n$ integral and the $MRIG_n$ laws). Section 5 considers the particular case of the STZ_n integral when the graph G associated to W is a tree. Then we generalize the STZ_n integral by computing in this case $\int_{C_W} \exp(-\langle x, y \rangle)(\det M_x)^{q-1} dx$ and thus, in particular, obtaining the norming constant for the density considered in Massam and Wesołowski [11]. Interestingly enough, this generalization allows us to not restrict to the case where the w_{ij}'s are non-negative. The reason is the not so well known fact: if the associated graph of a positive definite matrix $M = (m_{ij})$ is a tree then the symmetric matrix $M' = (\pm m_{ij})$ is still positive definite whatever the $\pm$ are outside of the diagonal; therefore C_W is unchanged. Section 6 mentions a striking consequence (Corollary 6.2) of the $MRIG_n$ integral: if $(B_1, \dots, B_n)$ is multivariate normal, i.e. $(B_1, \dots, B_n) \sim N(0, M_x)$, then

$$\Pr(B_1 > 0, \dots, B_n > 0) = \frac{1}{(2\pi)^{n/2}} \int_{C_W \cap \{t \le x\}} \frac{dt}{\sqrt{(x_1 - t_1) \cdots (x_n - t_n)} \sqrt{\det M_t}}.$$

Section 7 proves a marginal but delicate fact that the densities of the $MRIG_n$ distributions are continuous on the whole $\mathbb{R}^n$. A first version of this paper is on arXiv 1709.04843.

2. The $MRIG_n$ integral

2.1. The integral and its various forms

It is useful to recall a classical formula, which is in fact the particular case $n = 1$ of Theorem 2.2 and the starting point of an induction proof.

Lemma 2.1. *If $a > 0$ and $b \geq 0$ then*

$$\int_0^\infty e^{-\frac{a^2t^2}{2}-\frac{b^2}{2t^2}}\,dt = \sqrt{\frac{\pi}{2}}\frac{1}{a}e^{-ab}. \tag{5}$$

Various proofs of Lemma 2.1 exist in the literature. An elegant one considers the equivalent formulation

$$\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(at-\frac{b}{t})^2\right]dt = 1 \tag{6}$$

and proves (6) by the change of variable $x = \varphi(t) = t - \frac{b}{at}$ which preserves the Lebesgue measure on $\mathbb{R}$. This idea seems to be due to George Boole [6]. In the sequel, if a is a column vector, or more generally a matrix, then $a^\top$ denotes the transposed matrix of a.

Theorem 2.2. *Let $a_1, \ldots, a_n > 0$ and $b_1, \ldots, b_n \geq 0$. Then with $a = (a_1, \ldots, a_n)^\top$ and $b = (b_1, \ldots, b_n)^\top$ we have*

$$MRIG_n = \int_{C_W} e^{-\frac{1}{2}\,(a^\top M_x a + b^\top M_x^{-1} b)}\frac{dx}{\sqrt{\det M_x}} = \left(\frac{\pi}{2}\right)^{n/2}\frac{e^{-(a_1b_1+\cdots+a_nb_n)}}{a_1\times\cdots\times a_n} \tag{7}$$

Comments.

- Inserting $t = \sqrt{2x}$ in (5) we see that (5) is the particular case $n = 1$ of (7).
- Remarkably, the right hand side of (7) does not depend on W.
- Another presentation of (7) is

$$\int_{C_W}\exp(-\frac{1}{2}\,\|M_x^{1/2}a - M_x^{-1/2}b\|^2)\frac{dx}{\sqrt{\det M_x}} = \left(\frac{\pi}{2}\right)^{n/2}\frac{1}{a_1\times\cdots\times a_n}.$$

  For $n = 1$ this is nothing but (6) after the change of variable $x = t^2/2$.
- Another variation: from (23), writing for short $\sqrt{s} = (\sqrt{s_1}, \ldots, \sqrt{s_n})^\top$ we have

$$\left(\frac{2}{\pi}\right)^{n/2}\int_{C_W} e^{-\langle x,s\rangle-\frac{1}{2}b^\top M_x^{-1}b}\frac{dx}{\sqrt{\det M_x}} = \frac{1}{\sqrt{s_1\times\cdots\times s_n}}e^{-2\langle b,\sqrt{s}\rangle-\frac{1}{2}\sqrt{s}^\top W\sqrt{s}}.$$

- One more variation of (7) and (23) is obtained by considering a positive definite matrix $A = (a_{ij})_{1\leq i,j\leq n}$ in the formula

$$\left(\frac{2}{\pi}\right)^{n/2}\int_{C_W} e^{-\frac{1}{2}\,\mathrm{tr}\,(M_xA)-\frac{1}{2}b^\top M_x^{-1}b}\frac{dx}{\sqrt{\det M_x}} = \frac{1}{\sqrt{a_{11}\times\cdots\times a_{nn}}}e^{-(b_1\sqrt{a_{11}}+\cdots+b_n\sqrt{a_{nn}})-\frac{1}{2}\sum_{i,j=1}^n w_{ij}(\sqrt{a_{ii}a_{jj}}-a_{ij})}.$$

  If $A = \Sigma^{-1}$, consider the Gaussian random variable $X = (X_1, \ldots, X_n) \sim N(0, \Sigma)$. Recall that $\rho_{ij} = -a_{ij}/\sqrt{a_{ii}a_{jj}}$ is the correlation between X_i and X_j conditioned by all $(X_k; k \neq i, j)$. Therefore

$$\sum_{i,j=1}^n w_{ij}(\sqrt{a_{ii}a_{jj}}-a_{ij}) = \sum_{i,j=1}^n w_{ij}\sqrt{a_{ii}a_{jj}}(1+\rho_{ij}).$$

- In (23) the condition $a_1, \ldots, a_n > 0$ is easily relaxed to $a_1, \ldots, a_n \neq 0$: on the right hand side of $a_1, \ldots, a_n > 0$ replace a_i by $|a_i|$, Things are quite different for the condition $b_1, \ldots, b_n \geq 0$: see the comments of the example $n = 2$ in Section 3.

2.2. Proof of Theorem 2.2

Proof. We prove it by induction on n. As mentioned above, Lemma 2.1 is the case $n = 1$. Assume that the result is true for n. Consider

$$W^1 = \begin{bmatrix} W & c \\ c^\top & 0 \end{bmatrix},\; M^1 = \begin{bmatrix} M_x & -c \\ -c^\top & 2x_{n+1} \end{bmatrix}, \tag{8}$$

where $c=(c_1,\dots,c_n)^\top$ with $c_i\geq 0$ for all i. We now assume that $(x,x_{n+1})\in C_{W^1}$. From the positive definiteness of M^1 we see that the Schur complement $t^2=2x_{n+1}-c^\top M_x^{-1}c$ is positive. We write

$$M^1=\begin{bmatrix} I_n & 0\\ -c^\top M_x^{-1} & 1\end{bmatrix}\begin{bmatrix} M_x & 0\\ 0 & t^2\end{bmatrix}\begin{bmatrix} I_n & -M_x^{-1}c\\ 0 & 1\end{bmatrix}. \tag{9}$$

Equality (9) leads to the computation of $(M^1)^{-1}$ as follows:

$$(M^1)^{-1}=\begin{bmatrix} I_n & M_x^{-1}c\\ 0 & 1\end{bmatrix}\begin{bmatrix} M_x^{-1} & 0\\ 0 & t^{-2}\end{bmatrix}\begin{bmatrix} I_n & 0\\ c^\top M_x^{-1} & 1\end{bmatrix}=\begin{bmatrix} M_x^{-1}+t^{-2}M_x^{-1}cc^\top M_x^{-1} & t^{-2}M_x^{-1}c\\ t^{-2}c^\top M_x^{-1} & t^{-2}\end{bmatrix}.$$

Before writing down the integral $MRIG_{n+1}$ we observe that

$$\begin{aligned}(a^\top,a_{n+1})M^1\begin{pmatrix} a\\ a_{n+1}\end{pmatrix}&=a^\top M_x a-2a^\top c a_{n+1}+2x_{n+1}a_{n+1}^2\\ &=-2a^\top c a_{n+1}+a^\top M_x a+c^\top M_x^{-1}c a_{n+1}^2+t^2a_{n+1}^2\end{aligned}$$

$$\begin{aligned}(b^\top,b_{n+1})(M^1)^{-1}\begin{pmatrix} b\\ b_{n+1}\end{pmatrix}&=b^\top M_x^{-1}b+t^{-2}b^\top M_x^{-1}cc^\top M_x^{-1}b+2t^{-2}b^\top M_x^{-1}cb_{n+1}+t^{-2}b_{n+1}^2\\ &=b^\top M_x^{-1}b+t^{-2}(b_{n+1}+b^\top M_x^{-1}c)^2. \end{aligned}\tag{10}$$

Also observe that the convex set C_{W^1} is parameterized by (x,t) in $C_W\times(0,\infty)$ and that, from (9) we have $\det M^1=t^2\det M_x$. With this parameterization we have

$$\frac{dxdx_{n+1}}{\sqrt{\det M^1}}=\frac{dx}{\sqrt{\det M_x}}dt.$$

We now write $MRIG_{n+1}$ as follows

$$\begin{aligned}MRIG_{n+1}&=e^{a^\top ca_{n+1}}\int_{C_W}\exp-\frac12\left[a^\top M_x a+c^\top M_x^{-1}ca_{n+1}^2+b^\top M_x^{-1}b\right]\\ &\qquad\times\left(\int_0^\infty\exp-\frac12\left[t^2a_{n+1}^2+t^{-2}(b_{n+1}+b^\top M_x^{-1}c)^2\right]dt\right)\frac{dx}{\sqrt{\det M_x}}\end{aligned}$$

$$=\sqrt{\frac{\pi}{2}}\frac{1}{a_{n+1}}e^{a^\top ca_{n+1}-a_{n+1}b_{n+1}}\int_{C_W}\exp-\frac12\left[a^\top M_x a+(c^\top a_{n+1}+b^\top)M_x^{-1}(ca_{n+1}+b)\right]\frac{dx}{\sqrt{\det M_x}} \tag{11}$$

$$=\left(\frac{\pi}{2}\right)^{(n+1)/2}\frac{1}{a_1\times\cdots\times a_{n+1}}e^{-a^\top b-a_{n+1}b_{n+1}}. \tag{12}$$

In this chain of equalities (11) is a consequence of Lemma 2.1 applied to the pair

$$a_{n+1},\ b_{n+1}+b^\top M_x^{-1}c.$$

Here a comment is in order: a famous lemma of Stieltjes implies that M_x^{-1} has non-negative coefficients when $x\in C_W$. Let us detail the proof in this particular case: if $D=2\operatorname{diag}(x_1,\dots,x_n)$ then $M_x=D^{1/2}(I_n-A)D^{1/2}$ where $A=D^{-1/2}WD^{-1/2}$. Since M_x is positive definite, I_n-A is also positive definite. Now write $(I_n-A)^{-1}=I_n+A+\cdots+A^{2N-1}+A^N(I_n-A)^{-1}A^N$. Since $A^N(I_n-A)^{-1}A^N$ is positive semidefinite, its trace is ≥ 0 and therefore for all N

$$\sum_{k=0}^{2N-1}\operatorname{tr}(A^k)\leq\operatorname{tr}(I_n-A)^{-1}.$$

Since A has non-negative coefficients this implies that $\sum_{k=0}^\infty\operatorname{tr}(A^k)$ converges. In particular $\lim_{N\to\infty}\operatorname{tr}(A^{2N})=0$. This implies that all the eigenvalues of A are in $(-1,1)$ and therefore the series of matrices $S=\sum_{k=0}^\infty A^k$ converges to $(I_n-A)^{-1}$. Since A has non-negative coefficients the same is true for S and for $M_x^{-1}=D^{-1/2}SD^{-1/2}$. Furthermore, if the graph G has vertices $\{1,\dots,n\}$ and has edges $\{i,j\}$ present according to the fact that $a_{ij}>0$ or not, then $(I_n-A)^{-1}$ is positive definite if G is connected (this remark will be used in the proof of Lemma 2.4).

As a consequence $b_{n+1}+b^\top M_x^{-1}c\geq 0$ and therefore (5) is applicable. Equality (12) is a consequence of the induction hypothesis where the pair (a,b) is replaced by $(a,a_{n+1}c+b)$. The induction hypothesis is extended. □

2.3. Laplacian of W and parameterizations of C_W by $(0,\infty)^n$ and $\mathbb{R}^n$

In order to show in Section 2.4 that two other remarkable integrals can be deduced from the $MRIG_n$ integral (7), it is necessary to recall some definitions about Laplacian on graphs or weighted graphs (see for instance Bapat [1]). We define the Laplacian of W as the quadratic form on $\mathbb{R}^n$ defined by

$$v^\top L_W v = \sum_{i<j} w_{ij}(v_i - v_j)^2 = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n w_{ij}(v_i - v_j)^2. \tag{13}$$

If $s_i = \sum_{j=1}^n w_{ij}$ and if $D = \mathrm{diag}(s_1, \dots, s_n)$ the representative matrix of this quadratic form is $L_W = D - W$. From the definition it is semi positive definite, and since $(1, \dots, 1)^\top$ is a eigenvector of L_W associated to the eigenvalue zero, L_W cannot be positive definite. However, by adding a proper diagonal matrix

$$D_b = \mathrm{diag}(b_1, \dots, b_n)$$

with $b_i \geq 0$ the matrix $D_b + L_W$ can be positive definite. One can also remark that $W^1 = \begin{bmatrix} W & b \\ b^\top & 0 \end{bmatrix}$ implies that $L_{W^1} = \begin{bmatrix} D_b + L_W & -b \\ -b^\top & \sum_{j=1}^n b_j \end{bmatrix}$.

Lemma 2.3. *$D_b + L_W$ is positive definite if and only if for each connected component C of the graph associated to W there exists $k \in C$ such that $b_k > 0$.*

Proof. $\Leftarrow$ Enough is to assume that the associated graph is connected and that there exists a k such that $b_k > 0$. If v is such that $v^\top(D_b + L_W)v = 0$ then $v_k = 0$. Furthermore $v_i - v_j = 0$ if $w_{ij} > 0$. Since the associated graph is connected all the v_i's are equal, and they are zero like v_k : this shows the positive definiteness of $D_b + L_W$. $\Rightarrow$ Here again we can assume that G is connected. We have seen that if $b_i = 0$ for all i then $D_b + L_W = L_W$ cannot be positive definite. □

The next lemma describes an important parameterization of C_W by $(0, \infty)^n$. Note that it depends on a non-zero parameter $b = (b_1, \dots, b_n)^\top \in [0, \infty)^n$. The case $b = (1, \dots, 1)^\top$ is most useful.

Lemma 2.4. *Assume that the graph G associated to W is connected. Let $y \in (0, \infty)^n$, fix $b \in [0, \infty)^n$ such that $b \neq 0$ and define $x \in \mathbb{R}^n$ by*

$$2x_i = \frac{1}{y_i}\left(b_i + \sum_{j=1}^n w_{ij}y_j\right) \tag{14}$$

Then x belongs to C_W, we have $M_x = D_b D_y^{-1} + L_W$ and $y = M_x^{-1}b$, the map $y \mapsto x$ is a diffeomorphism from $(0, \infty)^n$ onto C_W and

$$dx = \frac{\det M_x}{2^n}\frac{dy}{y_1 \times \cdots \times y_n}. \tag{15}$$

Proof. We rewrite (14) as $2x_i y_i - \sum_{j=1}^n w_{ij}y_j = b_i$ and thus it is equivalent to $b = M_x y$. Denote

$$W^{(y)} = D_y W D_y \tag{16}$$

and $s_i^{(y)} = \sum_{j=1}^n w_{ij}y_i y_j = 2x_i y_i^2 - b_i y_i$. Therefore

$$D_{s^{(y)}} = 2D_y D_x D_y - D_b D_y, \quad L_{W^{(y)}} = D_{s^{(y)}} - W^{(y)} = 2D_y D_x D_y - D_b D_y - D_y W D_y.$$

From the definition (13) of the Laplacian we have $L_{W^{(y)}} = D_y L_W D_y$ and we get

$$D_b D_y + L_{W^{(y)}} = 2D_y D_x D_y - D_y W D_y, \quad D_b D_y^{-1} + L_W = M_x.$$

From Lemma 2.3 $M_x = D_b D_y^{-1} + L_W$ is positive definite and furthermore $y = M_x^{-1}b$. Equality $b = M_x y$ shows that the map $y \mapsto x$ from $(0, \infty)^n$ to C_W is injective since $0 = M_x(y - y')$ implies $y = y'$ from the definite positiveness of M_x. If $x \in C_W$, define $y = M_x^{-1}b$. The fact that M_x^{-1} has only non-negative coefficients implies that $y \in [0, \infty)^n$. The fact that G is connected implies that $y \in (0, \infty)^n$. We get $M_x = D_b D_y^{-1} + L_W$ and this shows the surjectivity since any $y \in (0, \infty)$ provides a positive definite matrix $D_b D_y^{-1} + L_W$. The fact that $y \mapsto x$ is a diffeomorphism from $(0, \infty)^n$ onto C_W is clear.

The differential of the map $y \mapsto x$ from C_W onto $(0, \infty)^n$ is

$$h \mapsto -2M_x^{-1}D_h M_x^{-1}b = -2M_x^{-1}D_h y. \tag{17}$$

For showing (17) we observe that the differential of $M \mapsto M^{-1}$ is $H \mapsto -M^{-1}HM^{-1}$ and that the differential of the map $x \mapsto M_x$ is $h \mapsto 2\,D_h$. The Jacobian of $y \mapsto x$ is therefore $\frac{2^n}{\det M_x} y_1 \times \cdots \times y_n$ and this proves (15). □

Replacing y_i by e^{t_i} we will use Lemma 2.4 in the next section under the following form:

Corollary 2.5. *Under the hypothesis of Lemma 2.4, for $t \in \mathbb{R}^n$ define*

$$2x_i(t) = b_i e^{-t_i} + \sum_{j=1}^{n} w_{ij} e^{t_j - t_i}.$$

Then the map $t \mapsto x = x(t)$ is a diffeomorphism from $\mathbb{R}^n$ onto C_W and $dx = \frac{\det M_x}{2^n} dt$.

2.4. The Disertori–Spencer–Zirnbauer and Disertori–Merkl–Rolles integrals

In application of Theorem 2.2 and Corollary 2.5, we prove two surprizing formulas DSZ_n and DMR_n due to Disertori, Spencer and Zirnbauer [8] and Disertori, Merkl and Rolles [7]. For describing them we need the following notations. We consider the quadratic form in (1.1) of the first paper:

$$v^\top D(t) v = \sum_{1\le i<j\le n} w_{ij} e^{t_i + t_j} (v_i - v_j)^2 + \sum_{k=1}^{n} b_k e^{t_k} v_k^2.$$

The element (i,i) of the corresponding $n \times n$ matrix $D(t)$ is $b_i e^{t_i} + \sum_{j=1}^{n} w_{ij} e^{t_i + t_j}$ and the off diagonal element (i,j) is $-w_{ij} e^{t_i + t_j}$. This is nothing but the quadratic form with matrix $D(t) = D_b D_y + L_{W(y)}$ as in (16) when $y_i = e^{t_i}$ for all i.

We introduce a function $G(t)$ which is only marginally different from the F defined by (1.2) in [8].

$$G(t) = \sum_{i<j} w_{ij}(\cosh(t_i - t_j) - 1) + \sum_{k=1}^{n} ((\cosh t_k - 1) b_k + t_k) \,. \tag{18}$$

With these notations, the surprising formula (1.4) of [8], see (19), is the subject of the following proposition.

Proposition 2.6. *Assume that W is such that the associated graph is connected and fix $b \in [0, \infty)^n$ with $b \neq 0$. Then*

$$DSZ_n = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-G(t)} \sqrt{\det D(t)} dt = 1. \tag{19}$$

Proof. In (7) we insert $a_1 = \cdots = a_n = 1$ and we make the change of variable $x \mapsto t$ from C_W onto $\mathbb{R}^n$ described in Corollary 2.5. We get

$$-\frac{1}{2}(a^\top M_{x(t)} a + b^\top M_{x(t)}^{-1} b) = -\frac{1}{2} \sum_{i=1}^{n} 2x_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} - \frac{1}{2} \sum_{i=1}^{n} b_i e^{t_i}$$

$$= -\frac{1}{2} \sum_{i=1}^{n} b_i (e^{t_i} + e^{-t_i}) - \frac{1}{2} \sum_{i<j} w_{ij} (e^{t_j - t_i} + e^{t_i - t_j} - 2) = -\sum_{i=1}^{n} b_i \cosh t_i - \sum_{i<j} w_{ij} (\cosh(t_j - t_i) - 1).$$

Since $D(t) = D_{y(t)} M_{x(t)} D_{y(t)}$ we have $\det D(t) = e^{\sum_{i=1}^{n} 2t_i} \det M_{x(t)}$. Using Corollary 2.5 we obtain (19). □

Similarly, formula (2.4) of Disertori, Merkl and Rolles [7] introduces a probability $\mu(ds, dt)$ on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\mu(ds, dt) = e^{-\frac{1}{2} s^\top D(t) s - G_1(t)} \det D(t) \frac{dtds}{(2\pi)^n} \tag{20}$$

where the function G_1 is quite close to the function G defined by (18) and is defined by

$$G_1(t) = \sum_{i<j} w_{ij}(\cosh(t_i - t_j) - 1) + \sum_{k=1}^{n} (e^{-t_k} b_k + t_k) = G(t) + \sum_{k=1}^{n} (1 - \sinh t_k) b_k.$$

If $(S, T) \sim \mu$ it is clear that S is Gaussian when conditioned by T. However, the fact that the total mass of μ is one is not that obvious. The result is stated in Proposition 2.7. We skip its proof which uses again Corollary 2.5. It is a consequence of the STZ_n integral (1), by the change of variable of Corollary 2.5. The hypotheses on W and b are the same as in Proposition 2.6.

Proposition 2.7. *Let $f(t) = \frac{1}{(\sqrt{2\pi})^n} e^{-G_1(t)} \sqrt{\det D(t)}$. Then f is a probability density on $\mathbb{R}^n$. Furthermore if $T \sim f(t)dt$ and $S|T \sim N(0, D(T)^{-1})$ then $(S, T) \sim \mu$ defined by (20).*

3. Examples

The following examples consider various graphs associated to W where some calculations about M_x are explicit.

3.1. The case $n = 2$.

We take

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ M_x = \begin{bmatrix} 2x_1 & -1 \\ -1 & 2x_2 \end{bmatrix},\ M_x^{-1} = \frac{1}{4x_1x_2 - 1}\begin{bmatrix} 2x_2 & 1 \\ 1 & 2x_1 \end{bmatrix}$$

and C_W is the convex set of $\mathbb{R}^2$ limited by one branch of a hyperbola

$$C_W = \{(x_1, x_2)\ ;\ x_1, x_2 > 0,\ 4x_1x_2 - 1 > 0\}.$$

Theorem 2.2 says that

$$\int\int_{C_W} \exp -[a_1^2x_1 + a_2^2x_2 - a_1a_2 + \frac{1}{4x_1x_2 - 1}(b_1^2x_2 + b_2^2x_1 + b_1b_2)]\frac{dx_1dx_2}{\sqrt{4x_1x_2 - 1}} = \frac{\pi}{2}\frac{e^{-a_1b_1-a_2b_2}}{a_1a_2}.$$

A warning: the extension of $MRIG_n$ to the case where some b_i' s are negative leads to a non elementary integral. The case $n = 2$ is appropriate for explaining this fact: following the steps of the proof of Theorem 2.2 we arrive up to a multiplicative constant to the integral

$$e^{a_1a_2}\int_0^{\infty} e^{-a_1^2x_1 - \frac{a_2^2+b_1^2}{4x_1} - a_2\left|b_2 + \frac{b_1}{2x_1}\right|}\frac{dx_1}{\sqrt{2x_1}}$$

that we cannot evaluate when $b_1b_2 < 0$.

3.2. The complete graph for $n \geq 3$

We consider the case where $w_{ij} = c$ for all $i \neq j$. Denote by J_n the $n \times n$ matrix with all entries equal to 1. Therefore $W = c(J_n - I_n)$.

Proposition 3.1. *If $W = c(J_n - I_n)$ then*

$$\det M_x = (c + 2x_1)\cdots(c + 2x_n)(1 - \sum_{i=1}^{n}\frac{c}{c + 2x_i}), \tag{21}$$

$$C_W = \{(x_1, \ldots, x_n); x_1, \ldots, x_n \geq 0,\ \sum_{i=1}^{n}\frac{c}{c + 2x_i} < 1\}. \tag{22}$$

Proof. Write $D = 2\mathrm{diag}(x_1, \ldots, x_n) + cI_n$. Therefore $M_x = D - cJ_n = D^{1/2}(I_n - A)D^{1/2}$ where $A = cD^{-1/2}J_nD^{-1/2} = vv^\top$ and

$$v = (\frac{\sqrt{c}}{\sqrt{c + 2x_1}}, \ldots, \frac{\sqrt{c}}{\sqrt{c + 2x_n}})^\top.$$

The eigenvalues of A are 0 with multiplicity $n - 1$ and $v^\top v = \sum_{i=1}^{n}\frac{c}{c+2x_i}$. This implies that the eigenvalues of $I_n - A$ are 1 with multiplicity $n - 1$ and $1 - \sum_{i=1}^{n}\frac{c}{c+2x_i}$. This leads to (21). To prove (22), clearly the right-hand side contains C_W. Conversely if $x_i > 0$ for all i writing $M_x = D^{1/2}(I - A)D^{1/2}$ shows $x \in C_W$ if and only if $I - A$ is positive definite, i.e. $1 - \sum_{i=1}^{n}\frac{c}{c+2x_i} > 0$. □

3.3. The daisy

We consider the case where

$$W = \begin{bmatrix} 0 & c_1 & c_2 & \ldots & c_n \\ c_1 & 0 & 0 & \ldots & 0 \\ c_2 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ c_n & 0 & 0 & \ldots & 0 \end{bmatrix},\ M_x = \begin{bmatrix} 2x_0 & -c_1 & -c_2 & \ldots & -c_n \\ -c_1 & 2x_1 & 0 & \ldots & 0 \\ -c_2 & 0 & 2x_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ -c_n & 0 & 0 & \ldots & 2x_n \end{bmatrix}$$

It is easy to see by induction that

$$\det M_x = 2^n x_1 \times \cdots \times x_n\left(2x_0 - \sum_{i=1}^{n}\frac{c_i^2}{2x_i}\right),$$

$$C_W = \{(x_0, \ldots, x_n); x_0, \ldots, x_n > 0,\ 2x_0 - \sum_{i=1}^{n}\frac{c_i^2}{2x_i} > 0\}.$$

It is elementary to write M_x^{-1} explicitly. If we write for simplicity

$$M(a,b,c)=\begin{bmatrix} a & c_1 & c_2 & \dots & c_n \\ c_1 & b_1 & 0 & \dots & 0 \\ c_2 & 0 & b_2 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ c_n & 0 & 0 & \dots & b_n \end{bmatrix}$$

$B=b_1\times\cdots\times b_n$, $D=\det M(a,b,c)=B\left(a-\sum_{i=1}^n \frac{c_i^2}{b_i}\right)$ and $H=M(a,b,c)^{-1}=(h_{ij})_{0\leq i,j\leq n}$ then for $i\neq j$ and distinct from 0 we have

$$h_{00}=\frac{B}{D},\ h_{0i}=-\frac{B}{D}\frac{c_i}{b_i},\ h_{ii}=\frac{1}{b_i}\left(1+\frac{B}{D}\frac{c_i^2}{b_i}\right),\ h_{ij}=\frac{c_ic_j}{b_ib_j}\left(1+\frac{B}{D}(\frac{c_i^2}{b_i}+\frac{c_j^2}{b_j})\right)$$

For $n=2$ it gives $D=\det M_x=8x_0x_1x_2-2x_2c_1^2-2x_1c_2^2$ and

$$M_x^{-1}=\frac{1}{D}\begin{bmatrix} 4x_1x_2 & 2c_1x_2 & 2c_2x_1 \\ 2c_1x_2 & 4x_0x_2-c_2^2 & c_1c_2 \\ 2c_2x_1 & c_1c_2 & 4x_0x_1-c_1^2 \end{bmatrix}$$

3.4. The chain

Define A_{n+1} as the graph $\overset{0}{\bullet}-\overset{1}{\bullet}-\overset{2}{\bullet}-\cdots-\overset{n}{\bullet}$ corresponding to the matrix

$$W=\begin{bmatrix} 0 & c_1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 0 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & 0 & c_3 & \cdots & 0 & 0 \\ 0 & 0 & c_3 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_n \\ 0 & 0 & 0 & 0 & \cdots & c_n & 0 \end{bmatrix}$$

where $c_1,\dots,c_n>0$. Thus $M_x=-W+\mathrm{diag}(2x_0,2x_1,\dots,2x_n)$ is a Jacobi matrix. Without losing generality we may assume that $c_1=\cdots=c_n=1$ by the transformation

$$\mathrm{diag}(\lambda_0,\lambda_1,\dots,\lambda_n)\,M_x\,\mathrm{diag}(\lambda_0,\lambda_1,\dots,\lambda_n)$$

where

$$\lambda_0=1,\ \lambda_1=\frac{1}{c_1},\ \lambda_{2p}=\frac{c_1c_3\dots c_{2p-1}}{c_2c_4\dots c_{2p}},\ \lambda_{2p+1}=\frac{c_2c_4\dots c_{2p}}{c_1c_3\dots c_{2p+1}}$$

thus replacing x_i by the affinity $x_i\lambda_i^2$. If $D_0=2x_0$ and $D_1=4x_0x_1-1$ then the determinant D_n of the matrix

$$M_x=\begin{bmatrix} 2x_0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2x_1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2x_2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2x_3 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & 0 & \cdots & 2x_{n-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2x_n \end{bmatrix}$$

is computable by the induction formula $D_n=2x_nD_{n-1}-D_{n-2}$. For instance for $n=3$ the set C_W is described by the four inequalities

$$x_0>0,\ 4x_0x_1-1>0,\ 4x_0x_1x_2-x_0-x_2>0,\ 16x_0x_1x_2x_3-4x_0x_3-4x_2x_3-4x_1x_2+1>0.$$

Since the chain A_{n+1} is also a tree, results of Section 5 are applicable to this example.

4. A study of the $MRIG_n$ distributions

4.1. $MRIG_n$ As a natural exponential family

Writing $s_i = a_i^2$ in (7) we obtain

$$\left(\frac{2}{\pi}\right)^{n/2}\int_{C_W} e^{-\langle x,s\rangle-\frac{1}{2}b^\top M_x^{-1}b}\frac{dx}{\sqrt{\det M_x}} = \frac{1}{\sqrt{s_1\times\cdots\times s_n}}e^{-(b_1\sqrt{s_1}+\cdots+b_n\sqrt{s_n})-\frac{1}{2}\sum_{i,j=1}^n w_{ij}\sqrt{s_is_j}}, \tag{23}$$

which suggests that the natural exponential family (NEF) concentrated on $C_W\subset\mathbb{R}^n$ generated by the unbounded measure

$$\mu(b,W)(dx) = e^{-\frac{1}{2}b^\top M_x^{-1}b}1_{C_W}(x)\frac{dx}{\sqrt{\det M_x}} \tag{24}$$

is interesting to study. For $n=1$ if $b_1>0$ this is nothing but a RIG distribution mentioned in (3) and if $b_1=0$ it is a Gamma family with shape parameter 1/2. For $n>1$ and $b=0$ this family is considered in Sabot, Tarrès and Zeng (2016). Given W and $a\in(0,+\infty)^n$, $b\in[0,+\infty)^n$ we consider the probability on $[0,+\infty)^n$ defined by

$$P(a;b,W)(dx) = (\frac{2}{\pi})^{n/2}\left(\prod_{j=1}^n a_je^{a_jb_j}\right)e^{-\frac{1}{2}a^\top M_xa-\frac{1}{2}b^\top M_x^{-1}b}\mathbf{1}_{C_W}(x)\frac{dx_1\times\cdots\times dx_n}{\sqrt{\det M_x}}.$$

We say that $P(a;b,W)$ is a $MRIG_n$ distribution. Theorem 2.2 proves that it is indeed a probability. From time to time we will use the notation $f(a;b,W)(x)$ for the density of $P(a;b,W)$. Note that $(X_1,\ldots,X_n)\sim P(a;b,0)$ iff $X_1,\ldots,X_n$ are independent and $X_k\sim RIG(a_k,\ b_k)$, $k\in\{1,\ldots,n\}$.

In this section, we show that if X has a $MRIG_n$ distribution then the subvector $(X_1,\ldots,X_k)$ has a $MRIG_k$ distribution. We also show that up to a translation factor, the conditional distribution of $(X_{k+1},\ldots,X_n)$ given $(X_1,\ldots,X_k)$ has a $MRIG_{n-k}$ distribution. Thus the class of $MRIG_n$ distributions has a remarkable property of stability by marginalization and conditioning. These facts have been independently observed by Sabot and Zeng [17] in their Lemma 5, and also mentioned in Sabot and Zeng [16] quoting the arXiv versions of Sabot and Zeng [16] and of the present paper.

We begin with the calculation of the Laplace transform of $P(a;b,W)$. Introducing the following function:

$$G(a;b,W) = (\frac{\pi}{2})^{n/2}\left(\prod_{j=1}^n\frac{e^{-a_jb_j}}{a_j}\right)e^{-\frac{1}{2}a^\top Wa}, \tag{25}$$

we remark that $P(a;b,W)(dx)$ can be written as

$$P(a;b,W)(dx) = \frac{1}{G(a;b,W)}e^{-(x_1a_1^2+\cdots+x_na_n^2)}\mu(b,W)(dx). \tag{26}$$

Under the form (26) we see that for fixed $b\in[0,\infty)^n$

$$F_b = \{P(a;b,W),\ a\in(0,\infty)^n\}$$

is a natural exponential family, parameterized by a and not by its natural parameter $(s_1,\ldots,s_n)=(a_1^2,\ldots,a_n^2)$, and generated by $\mu(b,W)$. From the fact that the mass of (26) is one, the Laplace transform of $\mu(b,W)$ is defined for $s\in(0,\infty)^n$ by

$$L_{\mu(b,W)}(s) = G((\sqrt{s_1},\ldots,\sqrt{s_n});b,W).$$

We deduce from this the form of the Laplace transform of $P(a;b,W)$ itself.

Proposition 4.1. *If $(X_1,\ldots,X_n)\sim P(a;b,W)$ then*

$$\mathbb{E}(e^{-(s_1X_1+\cdots+s_nX_n)}) = e^{\langle a,b\rangle-\langle\sqrt{a^2+s},b\rangle}e^{a^\top Wa-\sqrt{a^2+s}^\top W\sqrt{a^2+s}}\prod_{j=1}^n\frac{a_j}{\sqrt{a_j^2+s_j}}, \tag{27}$$

where we have written symbolically $\sqrt{a^2+s}=(\sqrt{a_1^2+s_1},\ldots,\sqrt{a_n^2+s_n})^\top$. In particular

$$\mathbb{E}(X_i) = m_i = \frac{1}{2a_i}\left(b_i+\sum_{j\neq i}w_{ij}a_j\right), \tag{28}$$

$$\mathbb{V}ar(X_i) = \frac{1}{4a_i^4}+\frac{m_i}{2a_i^2}, \tag{29}$$

$$\mathbb{C}ov(X_i,X_j) = -\frac{w_{ij}}{4a_ia_j}. \tag{30}$$

Comments. The one dimensional margins are the classical *RIG* distributions (3). More specifically the distribution of X_i is

$$RIG(a_i, b_i + \sum_{j=1}^{n} w_{ij}a_j) = RIG(a_i, 2a_i m_i).$$

In other terms the two parameters of the distribution of X_i's are the i components of the vectors a and $b + Wa$. Formula (28) expresses m_i with a formula which is the successful change of variable (14) where the pair (x, y) is replaced here by (m, a). Observe also that the covariance of (X_i, X_j) is never positive. It can be mentioned that, like for a Gaussian distribution, the parameters (a, b, W) of the distribution $MRIG_n$ are determined if we know the distributions of all pairs (X_i, X_j) : the knowledge of the distribution of X_i gives from (28) and (29) the knowledge of m_i and a_i. The knowledge of the distribution of (X_i, X_j) and of a gives from (30) the knowledge of w_{ij} and W, and then (28) gives the value of b_i. Estimation of the parameters can be designed from this remark. One more analogy with the Gaussian distributions is the fact that if $X \sim MRIG_n$ then X_i and X_j are independent if and only if they are uncorrelated: this can be read from the Laplace transform of X.

Proof of Proposition 4.1. Formula (27) comes immediately from

$$\mathbb{E}(e^{-s_1X_1-\cdots-s_nX_n}) = \frac{G(\sqrt{a_1^2+s_1}, \ldots, \sqrt{a_n^2+s_n}); b, W}{G(a; b, W)}.$$

Eqs. (28) and (29) are consequence of the properties of the one dimensional *RIG* given in (4). The simple formula (30) is obtained by $\mathbb{C}\mathrm{ov}(X_i, X_j) = \frac{\partial^2}{\partial s_i \partial s_j} \log G(\sqrt{a_1^2+s_1}, \ldots, \sqrt{a_n^2+s_n}); b, W|_{s=0}$. □

4.2. The marginals of the $MRIG_n$ distribution

For stating the next results we need the following notations:

- For vectors $(a_1, a_2, \ldots, a_n)^\top$ and $(b_1, b_2, \ldots, b_n)^\top$ we denote

  $$\tilde{a}_k = (a_1, \ldots, a_k)^\top, \ \tilde{b}_k = (b_1, \ldots, b_k)^\top.$$

  With this notation sometimes we write $P(\tilde{a}_n; \tilde{b}_n, W)$ for $P(a; b, W)$.
- If

  $$W_n = \begin{bmatrix} 0 & w_{12} & w_{13} & \ldots & w_{1n} \\ w_{12} & 0 & w_{23} & \ldots & w_{2n} \\ & & & \ddots & \\ \ldots & \ldots & \ldots & & \ldots \\ w_{1n} & w_{2n} & w_{3n} & \ldots & 0 \end{bmatrix}$$

  for $k \in \{2, 3, \ldots, n\}$ we take $c_k \in \mathbb{R}^{k-1}$ defined as the kth column of W_n but restricted to be above the diagonal, namely $c_k = (w_{1k}, w_{2k}, \ldots, w_{k-1,k})^\top$.
- If $k \in \{1, 2, 3, \ldots, n\}$ we write W_n by blocks as follows

  $$W_n = \begin{bmatrix} W_k & W_{k,n-k} \\ W_{k,n-k}^\top & W'_{n-k} \end{bmatrix}.$$

  In other terms $W_k = [w_{ij}]_{1\leq i,j\leq k}, \ W'_{n-k} = [w_{ij}]_{k+1\leq i,j\leq n}$.
- The killing symbol K from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ is defined by

  $$K(x_1, \ldots, x_n)^\top = (x_1, \ldots, x_{n-1})^\top.$$

  For instance $K\tilde{b}_k = \tilde{b}_{k-1}$. In general for $k < n$ we have

  $$K^{n-k}(x_1, \ldots, x_n)^\top = (x_1, \ldots, x_k)^\top.$$

Proposition 4.2. *If $(X_1, \ldots, X_n) \sim P(\tilde{a}_n, \tilde{b}_n, W)$ then $(X_1, \ldots, X_k) \sim P(\tilde{a}_k, B_k, W_k)$ where*

$$B_k = \tilde{b}_k + \sum_{j=k+1}^{n} a_j K^{j-k-1} c_j = \tilde{b}_k + W_{k,n-k}(a_{k+1}, \ldots, a_n)^\top.$$

Proof. For $k = n - 1$, this is claiming that $(X_1, \ldots, X_{n-1}) \sim P(\tilde{a}_{n-1}, \tilde{b}_{n-1} + a_n c_n, W_{n-1})$. Such a formula is essentially formula (11) when replacing n by $n + 1$.

We now prove the result by induction on $n-k$. Suppose that $(X_1,\dots,X_k)\sim P(\tilde{a}_k,B_k,W_k)$ is true. Then as for the passage from n to $n-1$ we can claim that $(X_1,\dots,X_{k-1})\sim P(\tilde{a}_{k-1},KB_k+a_kc_k,W_{k-1})$. Now we have

$$KB_k+a_kc_k=K\tilde{b}_k+a_kc_k+K\sum_{j=k+1}^n a_jK^{j-k-1}c_j=\tilde{b}_{k-1}+a_kc_k+\sum_{j=k+1}^n a_jK^{j-k}c_j=B_{k-1},$$

and the induction is extended. □

Comments.

- Proposition 4.2 could have been proved with the Laplace transform of Proposition 4.1, but seems that after all induction is simpler.
- A reformulation of Proposition 4.2 is the explicit form of the integral

$$\int_{C_{W'_{n-k}}} f_{\tilde{a}_n;\tilde{b}_n,W_n}(\tilde{x}_k,x_{k+1},\dots,x_n)dx_{k+1}\dots dx_n=f_{\tilde{a}_k,B_k,W_k}(\tilde{x}_k),$$

  namely

$$(\frac{2}{\pi})^{n/2}\left(\prod_{j=1}^n a_j\right)e^{\langle a,b\rangle-\frac{1}{2}a^\top Wa}\int_{C_{W'_{n-k}}} e^{-(x_1a_1^2+\dots+x_na_n^2)-\frac{1}{2}b^\top M_x^{-1}b}\mathbf{1}_{C_W}(x)\frac{dx_{k+1}\dots dx_n}{\sqrt{\det M_x}}$$
$$=(\frac{2}{\pi})^{k/2}\left(\prod_{j=1}^k a_j\right)e^{\langle\tilde{a}_k,B_k\rangle}e^{-\frac{1}{2}\tilde{a}_k^\top M_{\tilde{x}_k}\tilde{a}_k-\frac{1}{2}B_k^\top M_{\tilde{x}_k}^{-1}B_k}\mathbf{1}_{C_{W_k}}(\tilde{x}_k)\frac{1}{\sqrt{\det M_{\tilde{x}_k}}}.$$

- Inserting $b=0$ in Proposition 4.2 makes that $(X_1,\dots,X_n)$ has an STZ_n distribution. If we also take $k=n-1$ we see that $B_{n-1}=a_nc$ where $c=(w_{i,n})_{i=1}^{n-1}$. As a consequence, we see that any $MRIG_{n-1}$ distribution is a projection of some STZ_n distribution. This explains why Sabot, Tarrès and Zeng [15] indeed observe that one dimensional margins of an STZ_n distribution are *RIG* ones.

4.3. Conditional distributions under $MRIG_n$

Let us begin by some general observations about exponential families on a product $E\times F$ of two Euclidean spaces generated by the distribution $\pi(dx)K(x,dy)$. Let $\Theta\subset E\times F$ be the interior of the set

$$\{(t,s)\ ;\ L(t,s)=\int_{E\times F}e^{-\langle t,x\rangle-\langle s,y\rangle}\pi(dx)K(x,dy)<\infty\}.$$

Let us assume that Θ is the product of two open subsets of E and F respectively:

$$\Theta=\Theta_E\times\Theta_F, \tag{31}$$

let us fix $(t_0,s_0)\in\Theta$ and consider a random variable (X,Y) valued in $E\times F$ with density

$$\frac{1}{L(t_0,s_0)}e^{-\langle t_0,x\rangle-\langle s_0,y\rangle}\pi(dx)K(x,dy).$$

We are interested in the Laplace transform of the conditional distribution of $Y|X$. For computing this, consider the marginal density of X with respect to π :

$$\frac{1}{L(t_0,s_0)}\int_F e^{-\langle t_0,x\rangle-\langle s_0,y\rangle}K(x,dy)=e^{-\langle t_0,x\rangle}\frac{g(s_0;x)}{L(t_0,s_0)},$$

where we have introduced the auxiliary function

$$g(s_0;x)=\int_F e^{-\langle s_0,y\rangle}K(x,dy)$$

defined on $\Theta_F\times E$. As a consequence, the conditional distribution of $Y|X$ is $e^{-\langle s_0,y\rangle}K(X,dy)/g(s_0;X)$ and its Laplace transform is for $s+s_0\in\Theta_F$ the ratio

$$s\mapsto\frac{g(s+s_0;X)}{g(s_0;X)}. \tag{32}$$

Suppose now that we are able to identify a density on F having (32) as Laplace transform. In this case the problem of computation of the density of $Y|X$ will be solved.

This program will be applied to $E=\mathbb{R}^k$, $F=\mathbb{R}^{n-k}$, to a probability on $\mathbb{R}^n$ defined by its density

$$f(x)\propto e^{-\frac{1}{2}b^\top M_x^{-1}b}\mathbf{1}_{C_W}(x)\frac{1}{\sqrt{\det M_x}}$$

and finally to $t_0=(a_1^2,\dots,a_k^2)$, $s_0=(a_{k+1}^2,\dots,a_n^2)$. We prove in Section 7 that f is continuous on $\mathbb{R}^n$ when $b_1,\dots,b_n$ is not zero and when the graph associated to W is connected.

This condition (31) is fulfilled with $\Theta_E=(0,\infty)^k$ and $\Theta_F=(0,\infty)^{n-k}$. Of course $\tilde{X}_k=(X_1,\dots,X_k)$ and $(X_{k+1},\dots,X_n)$ replace X and Y. Also s is now $(s_{k+1},\dots .s_n)$ and $s+s_0$ is described by

$$A_k(s)=(\sqrt{a_{k+1}^2+s_{k+1}},\dots,\sqrt{a_n^2+s_n})^\top.$$

The crucial function $g(s_0;x)$ is now constructed from Proposition 4.2, where the marginal law of $(X_1,\dots,X_k)$ is computed. We get

$$g(a_{k+1}^2,\dots,a_n^2;\tilde{x}_k)=\frac{G(a;b,W)}{G(\tilde{a}_k;B_k,W_k)}\frac{e^{-\frac{1}{2}B_k^\top M_{\tilde{x}_k}^{-1}B_k}}{\sqrt{\det(M_{\tilde{x}_k})}}\mathbf{1}_{C_{W_k}}(\tilde{x}_k), \tag{33}$$

where the function $G(a;b,W)$ has been introduced in (25). Remember that the right hand side of (33) depends on $a_{k+1},\dots,a_n$ also through $B_k=\tilde{b}_k+W_{k,n-k}(a_{k+1},\dots,a_n)^\top$. Let us adopt the notation

$$B_k(s)=\tilde{b}_k+W_{k,n-k}A_k(s). \tag{34}$$

Here is now the Laplace transform of $(X_{k+1},\dots,X_n)$ given $\tilde{X}_k$

$$\mathbb{E}(e^{-s_{k+1}X_{k+1}-\cdots-s_nX_n}|\tilde{X}_k=\tilde{x}_k)=\frac{g((a_{k+1}^2+s_{k+1},\dots,a_n^2+s_n);\tilde{x}_k)}{g(a_{k+1}^2,\dots,a_n^2;\tilde{x}_k)}=PQR$$

$$P=\frac{G((\tilde{a}_k,A_k(s));b,W)}{G(a;b,W)},\quad Q=\frac{G(\tilde{a}_k;B_k,W_k)}{G(\tilde{a}_k;B_k(s),W_k)},\quad R=e^{-\frac{1}{2}B_k(s)^\top M_{\tilde{x}_k}^{-1}B_k(s)+\frac{1}{2}B_k^\top M_{\tilde{x}_k}^{-1}B_k}. \tag{35}$$

It is our intention to prove the existence of $\alpha=(\alpha_{k+1},\dots,\alpha_n)^\top$, $\beta=(\beta_{k+1},\dots,\beta_n)^\top$, $\gamma=(\gamma_{k+1},\dots,\gamma_n)^\top$ and of a matrix $\mathcal{W}$ such that

$$PQR=\frac{G(\sqrt{\alpha_{k+1}^2+s_{k+1}},\dots,\sqrt{\alpha_n^2+s_n};\beta,\mathcal{W})}{G(\alpha_{k+1},\dots,\alpha_n;\beta,\mathcal{W})}e^{-\gamma_{k+1}s_{k+1}-\cdots-\gamma_ns_n}$$

which is saying that the conditional distribution of $(X_{k+1}-\gamma_{k+1},\dots,X_n-\gamma_n)$ given $\tilde{X}_k$ is a $MRIG_{n-k}$ distribution. The next proposition gives the complete result:

Proposition 4.3. *For $X\sim P(a;b,W)$ consider $\alpha=(a_{k+1},\dots,a_n)^\top$, $\beta\in\mathbb{R}^{n-k}$ defined by*

$$\beta=(b_{k+1},\dots,b_n)^\top+W_{n-k,k}M_{\tilde{X}_k}^{-1}\tilde{b}_k,$$

the matrix $D=\mathrm{diag}(\gamma_{k+1},\dots,\gamma_n)$ defined as the diagonal part of the matrix $W_{n-k,k}M_{\tilde{X}_k}^{-1}W_{k,n-k}$ and

$$\mathcal{W}=W'_{n-k}+W_{n-k,k}M_{\tilde{X}_k}^{-1}W_{k,n-k}-D.$$

Then the conditional distribution of $(X_{k+1}-\gamma_{k+1},\dots,X_n-\gamma_n)$ given $\tilde{X}_k$ is $P(\alpha,\beta,\mathcal{W})$.

Proof. We have to analyse the dependency on s of the three quantities P,Q,R defined above by (35). However for simplification, we do not write the factors which do not depend on s. More specifically we introduce the following equivalence relation among non-zero functions f or g depending on s and possibly on other parameters like a, b, W by writing $f\equiv g$ if $f(s)/g(s)$ does not depend on s. For instance

$$P\equiv\prod_{j=k+1}^n(a_j^2+s_j)^{-\frac{1}{2}}\times e^{-(b_{k+1}\sqrt{a_{k+1}^2+s_{k+1}}+\cdots+b_n\sqrt{a_n^2+s_n})}e^{-\tilde{a}_k^\top W_{k,n-k}A_k(s)-\frac{1}{2}A_k(s)^\top W'_{n-k}A_k(s)},$$

$$Q\equiv e^{\tilde{a}_k^\top W_{k,n-k}A_k(s)},\quad R\equiv e^{-\frac{1}{2}B_k(s)^\top M_{\tilde{X}_k}^{-1}B_k(s)}\equiv e^{-A_k(s)^\top W_{n-k,k}M_{\tilde{X}_k}^{-1}\tilde{b}_k}\times e^{-\frac{1}{2}A_k(s)^\top W_{n-k,k}M_{\tilde{X}_k}^{-1}W_{k,n-k}A_k(s)}.$$

A patient analysis of the product PQR as a function of $A_k(s)$ gives Proposition 4.3. □

4.4. A convolution property of the $MRIG_n$ laws

The following proposition is a generalization of the following additive convolution:

$$RIG(a,b)*IG(a,b')=RIG(a,b+b'),$$

(see Barndorff–Nielsen and Koudou [3], Barndorff–Nielsen and Rydberg [4] and Barndorff–Nielsen, Blaesild and Seshadri [2]) with definitions in (2) and (3).

Proposition 4.4. *Let $a_i, b_i, b'_i > 0$ for $i \in \{1, \dots, n\}$. If $X = (X_1, \dots, X_n)$ has the $MRIG_n$ distribution $P(a; b, W)$, if $Y = (Y_1, \dots, Y_n)$ such that $Y_i \sim IG(a_i, b'_i)$ with independent components, and if X and Y are independent, then*

$$X + Y = (X_1 + Y_1, \dots, X_n + Y_n) \sim P(a; b + b', W).$$

Proof. Compute the Laplace transform, using Proposition 4.1 and (2). $\square$

4.5. Questions

Here are some unsolved problems linked to $MRIG_n$ laws:

- If $X \sim P(a; b, W)$ what is the distribution of M_X^{-1}? This random matrix is concentrated on a manifold of dimension n. This is a natural question since in one dimension if $X \sim RIG(a, b)$ then the distribution of $1/X$ is known and is $IG(b/2, 2a)$. However the Laplace transform of M_X^{-1}, namely $L(s) = \mathbb{E}(e^{-\mathrm{tr}\,(sM_X^{-1})})$ defined when s is a positive definite matrix of order n is not known in general. If $b = 0$ then X has an STZ_n distribution and Theorem 2.2 shows that $L(s)$ is known for s of rank one.
- Since in one dimension IG and RIG distributions are particular cases of the generalized inverse Gaussian laws, the natural extension of the $MRIG_n$ laws is to consider the probability densities on $\mathbb{R}^n$ proportional to
$$e^{-\frac{1}{2}\,a^\top M_x a - \frac{1}{2}\,b^\top M_x^{-1} b}(\det M_x)^{q-1}\mathbf{1}_{C_W}(x) \tag{36}$$
extending our familiar $MRIG_n$ integral from $1/2$ to an arbitrary real number q. But the corresponding integral extending Theorem 2.2 is untractable. However, in a particular case, namely if $b = 0$ and if the graph G associated to W is a tree, Proposition 5.1 computes the integral on C_W of the function (36). A related distribution has been analysed in Massam and Wesołowski [11] in connection with a multivariate version of the Matsumoto–Yor property (see e.g. Matsumoto and Yor [13], Letac and Wesołowski [10] and Massam and Wesołowski [12]).
- Probabilistic interpretations of the one dimensional laws IG and RIG are known, as hitting time and time of last visit of an interval $[a, \infty)$ by a drifted Brownian motion $t \mapsto mt + B(t)$ (in the respective cases $m > 0$ and $m < 0$). How to extend this to $MRIG_n$ laws? An answer to this problem is given in Sabot and Zeng [17] but one may look for other interpretations.

5. Another generalization of the Sabot–Tarrès–Zeng integral: the case of a tree

In this section we consider another generalization of a specialization of the Sabot–Tarrès–Zeng integral (1): we assume that the graph G associated to W is a tree but we replace in (1) the power $-1/2$ of $\det M_x$ by the real number $q-1 > -1$. Furthermore in Proposition 5.2, we are able to drop the restriction $w_{ij} \geq 0$ that we have done all along the paper, because of the following proposition of linear algebra:

Proposition 5.1. *Let $M = (m_{ij})_{1\leq i,j\leq n}$ be a symmetric matrix and let*

$$E = \{(i, j);\ 1 \leq i < j \leq n,\ \ m_{ij} \neq 0\}.$$

Assume that G is a graph with set of vertices $\{1, \dots, n\}$ and with E as set of edges. Then

(i) If G is a tree or a forest, $\det M$ is a polynomial in $(m_{ii})_{i=1}^n$ and in $(m_{ij}^2)_{(i,j)\in E}$;

(ii) if G is a tree or a forest, if M is positive definite and if $(\epsilon_{ij})_{1\leq i,j\leq n}$ is a symmetric matrix such that $\epsilon_{ij} = \pm 1$ and $\epsilon_{ii} = 1$ for all i, j then the symmetric matrix $(\epsilon_{ij} m_{ij})_{1\leq i,j\leq n}$ is also positive definite;

(iii) if the graph has a cycle then $\det M$ is a sum of monomials such that at least one of them contains an odd power of some m_{ij} with $i, j \in E$.

Comments. In general, changing the two off-diagonal entries m_{ij} and m_{ji} of a positive definite matrix M into $-m_{ij}$ and $-m_{ji}$ creates a new symmetric matrix which can be not positive definite anymore. The proposition shows that this is not the case when the graph associated to M is a tree. Part 3 shows that the fact that $\det M$ is a polynomial in $(m_{ij}^2)_{I,J} \in E$ characterizes the fact that the graph is a tree or a forest.

Proof. We prove (i) by induction on n. The result is clear when $n = 1$ and $n = 2$. Suppose that it is true for n and consider the case of a symmetric matrix M_1 of order $n + 1$ such that its associated graph G_1 is a tree. Without loss of generality, we may assume that $n + 1$ has only one neighbour in the tree and that this neighbour is n. This implies that M_1 has the form

$$M_1 = \begin{bmatrix} M & v \\ v^\top & m_{n+1,n+1} \end{bmatrix}, \quad v^\top = (0, \dots, 0, m_{n+1,n}),$$

where the symmetric matrix M of order n is associated to the graph G which is G_1 minus the vertex $n+1$. Since $n+1$ had n as the only neighbour, G is also a tree. Write also

$$M = \begin{bmatrix} M_{-1} & v_{-1} \\ v_{-1}^{\top} & m_{n,n} \end{bmatrix},$$

where M_{-1} is symmetric of order $n-1$. Assume that $m_{n+1,n+1} \neq 0$ and denote $c = m_{n+1,n}^2/m_{n+1,n+1}$ and

$$M' = \begin{bmatrix} M_{-1} & v_{-1} \\ v_{-1}^{\top} & m_{n,n} - c \end{bmatrix}.$$

Therefore we get that

$$\det M_1 = m_{n+1,n+1} \det M'.$$

Since M' is symmetric and since its associated graph is the tree G, the induction hypothesis implies that $\det M'$ is a polynomial with respect to the squares of the m_{ij} with $1 \leq i < j \leq n$ where (i,j) is an edge of G. Also, $\det M'$ is an affine function of c. Since c is a multiple of $m_{n+1,n}^2$, therefore the extension of the induction hypothesis to $n+1$ is done when $m_{n+1,n+1} \neq 0$. The extension to the case $m_{n+1,n+1} = 0$ is done by continuity of the polynomial $\det M_1$.

For showing (ii) we now apply (i) to the case where M is positive definite and we assume without loss of generality that G is a tree. We number its vertices $\{1,\ldots,n\}$ such that if G_k is the graph associated to the restriction M_k of M to $\{1,\ldots,k\}^2$, then G_k is a tree, a point which can be proved by induction. Denote $M_k(\epsilon) = (\epsilon_{ij} m_{ij})_{1\leq i,j\leq k}$. Since G_k is a tree and since M is positive definite, then $\det(M_k(\epsilon)) > 0$. From the theorem of principal determinants, $M_n(\epsilon)$ is positive definite.

For showing (iii) we assume first that G contains the cycle $1-2-\cdots-n-1$. We choose $m_{ij} = 0$ if $|i-j| \neq 1$ $m_{12} = m_{21} = a$ and $m_{i,j} = 1$ for the other edges of the cycle. With this choice the matrix M is

$$M_n = \begin{bmatrix} 0 & a & 0 & 0 & \cdots & 0 & 1 \\ a & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Standard techniques show that $\det M_n = \det M_{n+4}$ for $n \geq 3$ and that

$$\det M_3 = \det M_5 = 2a,\ \det M_4 = (a-1)^2,\ \det M_6 = -(a+1)^2.$$

Therefore one of the monomials is $\pm 2a$: this odd power is the one which was announced and this ends the proof of Proposition 5.1. □

For stating Proposition 5.2 we need to introduce the MacDonald function on $(0,\infty)$:

$$K_q(x) = \frac{1}{2}\int_0^{\infty} u^{q-1} e^{-\frac{1}{2}x(u+\frac{1}{u})} du.$$

It is useful to display a property of this integral

$$2\left(\frac{b}{a}\right)^q K_q(2ab) = \int_0^{\infty} v^{q-1} e^{-a^2 v - \frac{b^2}{v}} dv. \tag{37}$$

We denote by $s(i)$ the number of neighbours of i in the tree, namely the size of $\{j : w_{ij} > 0\}$.

Proposition 5.2. *Let $W = (w_{ij})_{1\leq i,j\leq j}$ be a symmetric matrix with zero diagonal such that its associated graph G is a tree. Let $M_x = 2\,\mathrm{diag}(x_1,\ldots,x_n) - W$ and let C_W be the set of x's such that M_x is positive definite. If $q > 0$ then*

$$\int_{C_W} e^{-\frac{1}{2}a^{\top} M_x a} (\det M_x)^{q-1} dx = 2^{q-1}\Gamma(q) e^{\frac{1}{2}a^{\top} W a} \prod_{i=1}^n a_i^{q(s(i)-2)} \prod_{i<j} |w_{ij}|^q K_q(a_i a_j |w_{ij}|). \tag{38}$$

Comments.

- For $y_1,\ldots,y_n > 0$ another presentation of (38) is

$$\int_{C_W} e^{-\langle x,y\rangle} (\det M_x)^{q-1} dx = 2^{q-1}\Gamma(q) \prod_{i=1}^n y_i^{\frac{1}{2}q(s(i)-2)} \prod_{i<j} w_{ij}^q K_q(\sqrt{y_i y_j} w_{ij}).$$

- Of course, inserting $q = 1/2$ gives back the Sabot–Tarrès–Zeng-integral in the case where G is a tree. To check this we use Lemma 2.1 which says

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}, \quad x > 0.$$

For $q = 3/2$ we use Watson [19] page 90 formula 12 for getting

$$K_{3/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\left(1 + \frac{1}{x}\right), \quad x > 0,$$

and we obtain

$$\int_{C_W} e^{-\frac{1}{2}a^\top M_x a}\sqrt{\det M_x}dx = \left(\frac{\pi}{2}\right)^{n/2}\prod_{i=1}^n a_i^{-3}\prod_{i<j}(1 + a_i a_j w_{ij}).$$

- We give a proof of Proposition 5.1, while another proof could be extracted from Massam and Wesołowski [11], where the authors consider the NEF generated by the unbounded measure

$$1_{C_W}(x)(\det M_x)^{q-1}dx$$

and independence properties of distributions from this NEF. Bobecka [5] has a multivariate generalization.

Proof. We proceed by induction on n. This is correct for $n = 1$ since in this case $s(1) = 0$ and since the empty product $\prod_{i<j}$ is one. Suppose that the formula (38) is true for n and let us extend it to $n + 1$. We use the same notation as in Section 2.2: we keep the notations a, W and M_x for the matrices of order n as before and we consider the block matrices M^1 and W^1 defined by (8). We now use a different factorization of M^1 by writing

$$M^1 = \begin{bmatrix} I_n & -\frac{c}{2x_{n+1}} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} M_x - \frac{cc^\top}{2x_{n+1}} & 0 \\ 0 & 2x_{n+1} \end{bmatrix}\begin{bmatrix} I_n & 0 \\ -\frac{c^\top}{2x_{n+1}} & 1 \end{bmatrix}. \tag{39}$$

Since the graph G^1 which is associated to W^1 is a tree, without loss of generality we assume that the vertex $n + 1$ has only one neighbour which is n. In other terms, we may assume that the vector c of $\mathbb{R}^n$ has the form

$$c = (0, \ldots, 0, w_{n,n+1})^\top.$$

This choice implies also that the graph G associated to W is still a tree. Formula (39) implies that C_{W^1} is the set of $(x, x_{n+1}) \in \mathbb{R}^{n+1}$ such that $x_{n+1} > 0$ and such that the diagonal y of the matrix

$$M_y = M_x - \frac{cc^\top}{2x_{n+1}} = M_x - \begin{bmatrix} 0 & 0 \\ 0 & \frac{w^2_{n,n+1}}{2x_{n+1}} \end{bmatrix},$$

namely $y = (x_1, \ldots, x_{n-1}, x_n - \frac{w^2_{n,n+1}}{4x_{n+1}})^\top$, belongs to C_W. The Jacobian of the transformation $(x, x_{n+1}) \mapsto (y, x_{n+1})$ is one. Therefore we can write

$$\begin{aligned}
&\int_{C_{W^1}} e^{-\frac{1}{2}a^\top M_x a + a^\top c a_{n+1} - a^2_{n+1}x_{n+1}}(\det M^1)^{q-1}dxdx_{n+1} \\
&= e^{a^\top c a_{n+1}}\int_{C_{W^1}} e^{-\frac{1}{2}a^\top(M_x - \frac{cc^\top}{2x_{n+1}})a - \frac{(c^\top a)^2}{4x_{n+1}} - a^2_{n+1}x_{n+1}}\det(M_x - \frac{cc^\top}{2x_{n+1}})^{q-1}(2x_{n+1})^{q-1}dxdx_{n+1} \\
&= \left(\int_{C_W} e^{-\frac{1}{2}a^\top M_y a}(\det M_y)^{q-1}dy\right) \times \left(e^{a^\top c a_{n+1}}\int_0^\infty e^{-\frac{(c^\top a)^2}{4x_{n+1}} - a^2_{n+1}x_{n+1}}(2x_{n+1})^{q-1}dx_{n+1}\right).
\end{aligned}$$

The latter integral is expressed with (37) as

$$e^{|w_{n,n+1}|a_n a_{n+1}}\frac{a_n^q}{a_{n+1}^q}|w_{n,n+1}|^q K_q(a_n a_{n+1}|w_{n,n+1}|),$$

and the former one is (38), from the induction hypothesis. To conclude, observe that the number of neighbours of $n + 1$ in G^1 is one, and that the number of neighbours of n in G^1 is the number $s(n)$ of neighbours of n in G plus one. □

6. If $B \sim N(0, M_x)$ what is $\Pr(B_1 > 0, \ldots, B_n > 0)$?

Of course the exact solution of this question cannot be found. However for any $x \in C_W$ denote by

$$f(x) = \Pr(B_1 > 0, \ldots, B_n > 0).$$

Then formula (23) enables us to compute the Laplace transform of $f(x)\mathbf{1}_{C_W}(x)$. The trick is to observe that the first member of (23) involves the density $g_x(b)$ of $N(0, M_x)$ when the $b_1, \dots, b_n$ are restricted to be > 0. Of course $f(x) = \int_{\mathbb{R}^n_+} g_x(b)db$, and $b \mapsto g_x(b)/f(x)$ is a probability density on $\mathbb{R}^n_+$.

One can even get a knowledge of the Laplace transform of $b \mapsto g_x(b)$. More specifically

Proposition 6.1. *For $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n_+$ and $x \in C_W$, denote $\int_{\mathbb{R}^n_+} e^{-\langle \theta, b\rangle} g_x(b)db = f(x, \theta)$. Then for $y = (y_1, \dots, y_n) \in \mathbb{R}^n_+$ we have*

$$\int_{C_W} e^{-\langle x,y\rangle} f(x,\theta)dx = \frac{1}{2^n \sqrt{y_1(y_1+\theta_1)} \dots \sqrt{y_n(y_n+\theta_n)}} e^{-\frac{1}{2}\sum_{i,j=1}^n w_{ij}\sqrt{y_i y_j}}. \tag{40}$$

In particular

$$\int_{C_W} e^{-\langle x,y\rangle} f(x)dx = \frac{1}{2^n y_1 \times \dots \times y_n} e^{-\frac{1}{2}\sum_{i,j=1}^n w_{ij}\sqrt{y_i y_j}}.$$

Proof. Enough is to multiply both sides of (23) by $e^{-\langle \theta, b\rangle}$ and integrate with respect to b on $\mathbb{R}^n_+$. Permuting the integrations on the left hand side leads to (40). □

Corollary 6.2.

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{C_W \cap \{t \le x\}} \frac{dt}{\sqrt{(x_1-t_1)\dots(x_n-t_n)}\sqrt{\det M_t}}. \tag{41}$$

Proof. Denote

$$h(x) = \frac{1}{\pi^{n/2}(x_1 \times \dots \times x_n)^{1/2}} \mathbf{1}_{\mathbb{R}^n_+}(x), \quad g(x) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\sqrt{\det M_x}} \mathbf{1}_{C_W}(x).$$

Consider the Laplace transforms $L_f(y)$, $L_g(y)$, $L_h(y)$ defined for $y_1, \dots, y_n > 0$. They are respectively given by (40) with $\theta = 0$, by the Sabot–Tarrès–Zeng integral (1) and by

$$\int_{\mathbb{R}^n_+} e^{-\langle x,y\rangle} h(x) = \frac{1}{\sqrt{y_1 \times \dots \times y_n}}.$$

As a consequence $L_f = L_g L_h$ which implies that f is the convolution product of g and h and proves (41). □

Corollary 6.3. *With the notation $D_y = \mathrm{diag}(y_1, \dots, y_n)$ and $y = M_x^{-1}b$, where $b = (1, \dots, 1)^\top$, we have*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \sqrt{y_1 \times \dots \times y_n} \int_{(0,\infty)^n} \left(\frac{\det(D^{-1}_{u+y} + L_W)}{\prod_{i=1}^n (u_i(u_i+y_i))} \right)^{1/2} du_1 \dots du_n \tag{42}$$

with the Laplacian L_W defined in (13).

Proof. In (41) we make the change of variable introduced in Lemma 2.4, namely $s = M_t^{-1}b$. We also observe that again from Lemma 2.4 it follows that $M_x - M_t = D_b(D_y^{-1} - D_s^{-1})$ and that

$$\prod_{i=1}^n (x_i - t_i) = \frac{1}{2^n}\det(M_x - M_t) = \frac{1}{2^n}\det D_b(D_y^{-1} - D_s^{-1}) = \frac{1}{2^n}\prod_{i=1}^n b_i \frac{s_i - y_i}{s_i y_i}.$$

Using the fact that $b = (1, \dots, 1)^\top$ we get

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{y_1}^{\infty} \dots \int_{y_n}^{\infty} \left(\det(D_s^{-1} + L_W) \prod_{i=1}^n \frac{y_i}{s_i(s_i - y_i)} \right)^{1/2} ds_1 \dots ds_n.$$

Hence the change of variables: $u_i = s_i - y_i$, $i \in \{1, \dots, n\}$, yields (42). □

Comments.

- Applying formula (41) even to the case $n = 2$ is surprising, since the left hand side is explicitly known: recall that if

$$(X_1, X_2) \sim N\left(0, \begin{bmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{bmatrix}\right) \Rightarrow \Pr(X_1 > 0, X_2 > 0) = \frac{\alpha}{2\pi}.$$

Therefore, if $M_x = \begin{bmatrix} 2x_1 & -w \\ -w & 2x_2 \end{bmatrix}$ formula (41) gives the following double integral on the domain $D = \{(t_1, t_2),\ \ t_1 < x_1,\ t_2 < x_2,\ w < 2\sqrt{t_1t_2}\}$:

$$\arccos \frac{w}{2\sqrt{x_1x_2}} = \int_D \frac{dt_1dt_2}{\sqrt{(x_1 - t_1)(x_2 - t_2)(4t_1t_2 - w^2)}},$$

an identity not so easy to check directly.

- Some comments about tentative applications to Bayesian analysis of $MRIG_n$ are in order. Recall that a positive matrix $A = \rho I_n - C$ is called a M-matrix if $C = (c_{ij})_{1\leq i,j\leq n}$ is such that $c_{ij} \geq 0$ for all i, j. Of course with our usual notation and for $x \in C_W$ then M_x is a M-matrix: we have just to define $c_{ij} = w_{ij}$ for $i \neq j$, $\rho = \max_i 2x_i$ and $c_{ii} = \rho - 2x_i$ for seeing this fact. The M-matrices are widely used in statistics since for $X \sim N(0, \Sigma)$ then the density $g(x)$ of X has the MTP_2 property, namely for all $x, y \in \mathbb{R}^n$

$$g(\min(x_1, y_1)\ldots, \min(x_n, y_n))g(\max(x_1, y_1)\ldots, \max(x_n, y_n)) \geq g(x)g(y)$$

if and only if Σ^{-1} is a M-matrix: we refer for instance to Karlin and Rinott [9] Page 482 for this fact. In Theorem 3 of the same paper it is proved that for $X \sim N(0, \Sigma)$ and for all i, j the covariance of X_i, X_j conditioned by $\{X_k;\ 1 \leq k \leq n,\ k \neq i, j\}$ is non negative if and only if Σ^{-1} is a M-matrix. From the point of view of Bayesian analysis two types of Gaussian models come to mind

1. $\{N(0, M_\theta^{-1})\ ;\ \theta \in C_W\}$. If $X \sim N(0, M_\theta^{-1})$ its density is

$$\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}x^\top M_\theta x} \sqrt{\det M_\theta}.$$

The densities have the MTP_2 property and the conditional covariances are all non negative. In order to use the $MRIG_n$ integral one is tempted to consider the a priori measure

$$\pi(d\theta) = e^{-\frac{1}{2}b^\top M_\theta^{-1}b} 1_{C_W}(\theta) \frac{d\theta}{\det M_\theta},$$

which is unfortunately unbounded since $x \mapsto \int_{C_W} N(0, M_\theta)(x)\pi(d\theta)$ is an unbounded density. From (23) and the last comment before the proof of Theorem 2.2, this density is proportional to $\prod_{i=1}^n e^{-a_i|x_i|}/|x_i|$.

2. $\{N(0, M_\theta)\ ;\ \theta \in C_W\}$. If $X \sim N(0, M_\theta)$ its density is

$$\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}x^\top M_\theta^{-1}x} \frac{1}{\sqrt{\det M_\theta}}.$$

These densities have less attractive properties from the MTP_2 point of view. Nevertheless the a priori measure

$$\pi(d\theta) = e^{-(a_1^2\theta_1+\cdots+a_n^2\theta_n)} 1_{C_W}(\theta)d\theta$$

is bounded. However a major defect of this choice is the fact that $x \mapsto \int_{C_W} N(0, M_\theta^{-1})(x)\pi(d\theta)$ is computable (by (23)) only if $x_1, \ldots, x_n$ are all non negative (again, see example $n = 2$ in Section 3).

7. Continuity of the density of the $MRIG_n$ laws

Proposition 7.1. *If $b_1, \ldots, b_n \geq 0$ with $b \neq 0$ and if the graph associated to W is connected then the function*

$$f(x) = e^{-\frac{1}{2}b^\top M_x^{-1}b} \frac{1}{\sqrt{\det M_x}} 1_{C_W}(x)$$

is continuous on $\mathbb{R}^n$.

Proof. The continuity of f is clear outside of the boundary of C_W, namely outside of the set ∂C_W of $x \in \mathbb{R}^n$ such that M_x is positive semidefinite with $\det M_x = 0$. In the sequel we fix $x \in \partial C_W$ and we prove the continuity of f at this point x.

First step. We show that if $t = (t_1, \ldots, t_n)^\top \in \mathbb{R}^n$ is such that $M_xt = 0$ and if $t_{i_0} > 0$ for some i_0, then $t_i > 0$ for all $i = 1, \ldots, n$. To see this, we use the notation $t_i^+ = \max(0, t_i)$, $t_i^- = t_i^+ - t_i$ and

$$t^+ = (t_1^+, \ldots, t_n^+)^\top, \quad t^- = t^+ - t.$$

Since $0 = M_xt = M_xt^+ - M_xt^-$ we multiply by $(t^+)^\top$ on the left for getting $(t^+)^\top M_xt^+ = (t^+)^\top M_xt^-$. Since $t_i^+t_i^- = 0$ we have that $(t^+)^\top M_xt^- \leq 0$. Since M_x is positive semidefinite we have that $(t^+)^\top M_xt^+ = 0$ and therefore $M_xt^+ = 0$. Without loss of generality, assume that $t^+ = (t_1, \ldots, t_k, 0, \ldots, 0)$ with $t_1, \ldots, t_k > 0$. Let us show that $k = n$. Since $t_{i_0} > 0$ we have $k > 0$. Suppose that $k < n$. We now split M_x in blocks

$$M_x = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix},$$

where A is a (k, k) matrix. Clearly since $M_x t^+ = 0$ we get $B^\top(t_1, \dots, t_k)^\top = 0$. Since it holds for all $t_j > 0$ for $j \in \{1, \dots, k\}$ this implies that $B = 0$. This contradicts the fact that G is connected, and finally $k = n$.

Second step. We show that no principal minor of M_x of order $n-1$ can be zero. Suppose for instance that the cofactor $C_{i_0}(x)$ of $2x_{i_0}$ is zero. This implies that there exists a non-zero $t \in \mathbb{R}^n$ such that $M_x t = 0$ and $t_{i_0} = 0$. From the first step, this is impossible.

Third step. Consider a sequence $(x_k)_{k=1}^\infty$ in C_W converging to x and let us show that $f(x_k)$ converges to zero. This is equivalent to show that

$$E_k = b^\top M_{x_k}^{-1} b + \log\det M_{x_k} \to \infty.$$

Recall that we have assumed that there exists i_0 such that $b_{i_0} > 0$. Recall also that all coefficients of $M_{x_k}^{-1}$ are non-negative. As a consequence

$$E_k \geq b_{i_0}^2 C_{i_0}(x_k) \frac{1}{\det M_{x_k}} + \log\det M_{x_k}.$$

As polynomials in x_k we have that $C_{i_0}(x_k)$ converges to $C_{i_0}(x)$ and $\det M_{x_k}$ converges to $\det M_x$. Since $C_{i_0}(x_k) > 0$ from the second step and since $\det M_x = 0$, we have shown that E_k tends to infinity and the proof is done. □

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