CALCULUS III

Sample exercises (part II), 2012

Surface integrals of the second type

1. Applying Stokes' formula, transform the integrals:

   (a) $\oint\limits_C (x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz$;

   (b) $\oint\limits_C y\,dx + z\,dy + x\,dz$;

2. Applying Stokes' formula, find the given integrals and verify the results by direct calculations:

   (a) $\oint\limits_C (y + z)\,dx + (z + x)\,dy + (x + y)\,dz$, where C is the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 0$;

   (b) $\oint\limits_C (y - z)\,dx + (z - x)\,dy + (x - y)\,dz$, where C is the ellipse $x^2 + y^2 = 1$, $x + z = 1$;

   (c) $\oint\limits_C x\,dx + (x + y)\,dy + (x + y + z)\,dz$, where C is the curve $x = 2\sin t$, $y = 2\cos t$, $z = 2(\sin t + \cos t)$ $(0 \le t \le 2\pi)$;

   (d) $\oint\limits_{ABCA} y^2\,dx + z^2\,dy + x^2\,dz$, where $ABCA$ is the contour of the triangle ABC with vertices $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$;

   (answer: (a) 0; (b) 4π; (c) $-\pi a^2$; (d) $-a^3$)

3. Using Stokes' theorem compute the circulation (the work of the vector field) of the vector $a = x^2y^3\vec{i} + \vec{j} + z\vec{k}$ along the circumference $x^2 + y^2 = R^2$, $z = 0$, taking the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$ for the surface.

4. Applying the Ostrogradsky-Gauss formula, transform the following surface integrals over the closed surfaces S bounding the volume V ($\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of the outer normal to the surface S):

   (a) $\iint\limits_S xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx$;

(b) $\iint\limits_S \frac{x\cos\alpha + y\cos\beta + z\cos\gamma}{\sqrt{x^2+y^2+z^2}}\,dS.$

5. Using the Ostrogradsky-Gauss formula, compute the following surface integrals:

   (a) $\iint\limits_S z^2\,dx\,dy + x^2\,dy\,dz + y^2\,dz\,dx$, where S is the external side of the surface of the cube $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$;

   (b) $\iint\limits_S z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$, where S is the external side of a pyramid bounded by the surfaces $x+y+z=3$, $x=0$, $y=0$, $z=0$;

   (c) $\iint\limits_S z^3\,dx\,dy + x^3\,dy\,dz + y^3\,dz\,dx$, where S is the external side of the sphere $x^2+y^2+z^2=9$;

   (answer: (a) 48; (b) $\frac{27}{2}$; (c) $\frac{12}{5}\pi 3^5$)

6. Find the flux of the vector

   (a) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ through the total surface of the cylinder $x^2+y^2 \le 1$, $0 \le z \le 2$;

   (b) $\vec{r} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ through

   - the lateral surface of the cone $4(x^2+y^2) \le z^2$, $0 \le z \le 2$,
   - the total surface of the cone.

   (answer: (a) 6π; (b) $\frac{11}{5}\pi$, $\frac{27}{5}\pi$)

7. Evaluate $\mathrm{div}\left(\frac{\vec{r}}{\|r\|_2}\right)$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\|r\|_2$ is the Euclidean norm of $\vec{r}$, i.e. $\|r\|_2 = \sqrt{x^2+y^2+z^2}$.

8. Evaluate the divergence and the rotation of the vector:

   (a) $\vec{r}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$,

   (b) $f(r)\vec{c}$, where $r = \sqrt{x^2+y^2+z^2}$ and $\vec{c}$ is a constant vector.

9. Compute $\mathrm{div}\,(\mathrm{grad} U)$.

10. Compute $\mathrm{rot}\,(\mathrm{grad} U)$ and $\mathrm{div}\,(\mathrm{rot}\vec{a})$.

    (answer: 0; 0)

11. Compute length and direction of the vector $\mathrm{rot}\vec{a}$ at the point $(1, 2, -2)$, where $\vec{a} = \frac{y}{z}\vec{i} + \frac{z}{x}\vec{j} + \frac{x}{y}\vec{k}$.

Series

12. Expand the following functions in positive integral powers of x, find the intervals of convergence of the resulting series and investigate the behaviour of their remainders:

    (a) a^x $(a > 0)$,

    (b) $\sin(x + \frac{\pi}{4})$,

    (c) $\cos x$,

    (d) $\sin^2 x$.

13. Expand the following functions in positive integral powers of x and indicate the intervals of convergence of the resulting series:

    (a) $\dfrac{2x-3}{(x-1)^2}$,

    (b) e^{x^2},

    (c) $\sinh x$,

    (d) $\cos 2x$,

    (e) $\ln \dfrac{1+x}{1-x}$

    (f) $(1 + e^x)^3$,

    (g) $\sqrt[3]{8 + x}$.

    (answers: intervals of convergence: (a) $|x| < 1$; (b) $-\infty < x < \infty$; (c) $-\infty < x < \infty$; (e) $|x| < 1$; (f) $-\infty < x < \infty$; (g) $-\infty < x < \infty$)

14. Write the first three nonzero terms of the expansion of the following functions in powers of x:

    (a) $\tan x$,

    (b) $\tanh x$,

    (c) $e^{\cos x}$.

15. Expand the function $x^3 - 2x^2 - 5x - 2$ in a series of powers of $x + 4$.

16. Expand the function $\ln x$ in a series of powers of $x - 1$.

17. What is the magnitude of the error if we put approximately

$$e \approx 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}?$$

    (answer: $|E| < \frac{e}{5!} < \frac{1}{40}$)

18. To what degree of accuracy will we calculate the number $\frac{\pi}{4}$, if we make use of the series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots,$$

by taking the sum of its first five terms when $x = 1$?

(answer: $|E| < \frac{1}{11}$)

19. How many terms do we have to take of the series

$$\cos x = 1 - \frac{x^2}{2!} + \dots,$$

in order to calculate $\cos 18^\circ$ to three decimal places?

(answer: two terms)

20. Calculate $\sqrt[3]{7}$ to two decimals by expanding the function $\sqrt[3]{8+x}$ in a series of powers of x.

(answer: 1.92)

21. Evaluate $\displaystyle\int_0^{1/2} \frac{\sin x}{x}\, dx$ to four decimals.

(answer: $\frac{1}{2} - \frac{1}{2^3 \cdot 3 \cdot 3!} \approx 0.4931$)

22. Evaluate $\displaystyle\int_0^{1} e^{-x^2}\, dx$ to four decimals.

(answer: 0.7468)

23. Expand the following functions in a Fourier series in the interval $(-\pi, \pi)$, determine the sum of the series at the points of discontinuity and at the end-points of the interval:

(a) $f(x) = \begin{cases} -1 & \text{when} \quad -\pi < x \leq 0, \\ 1 & \text{when} \quad 0 < x \leq \pi; \end{cases}$

(b) $f(x) = x^2$;

(c) $f(x) = \sin ax$, $a \in \mathbb{R}$.

24. Expand the function $f(x) = x^2$, in the interval $(0, \pi)$, into incomplete Fourier series: a) of sines of multiple arcs, b) of cosines of multiple arcs. Find the sums of the following number series by means of the expansion obtained:

(a) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$;

(b) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$.

25. Expand the function $f(x) = \begin{cases} x & \text{when} \quad 0 < x \le \frac{\pi}{2}, \\ 0 & \text{when} \quad \frac{\pi}{2} < x < \pi; \end{cases}$, in the interval $(0, \pi)$, in sines of multiple arcs.

26. Expand the following functions, in the indicated intervals, in incomplete Fourier series: a) in sines of multiple arcs, b) in cosines of multiple arcs:

   (a) $f(x) = 1, \quad x \in [0, 1];$
   (b) $f(x) = x, \quad x \in [0, a];$
   (c) $f(x) = x^2, \quad x \in [0, 2\pi].$