CALCULUS III

Sample exercises (part I), 2013

Line integrals

1. Evaluate the following integrals:

   (a) $\int\limits_C (x+y)\,ds$, where C is a triangle with vertices $A(0,0)$, $B(1,0)$, $C(0,1)$,

   (b) $\int\limits_C (x^2+y^2)\,ds$, where C is given by $x=\cos t+t\sin t$, $y=\sin t-t\cos t$, $t\in[0,2\pi]$,

   (c) $\int\limits_C \sqrt{x^2+y^2}\,ds$, where C is the contour $x^2+y^2=2x$,

   (d) $\int\limits_C (x^2+y^2+z^2)\,ds$, where C is given by $x=\cos t$, $y=\sin t$, $z=t$, $t\in[0,2\pi]$,

   (e) $\int\limits_C x^2\,ds$, where C is a circle given by intersection of $x^2+y^2+z^2=1$ with $x+y+z=0$,

   (f) $\int\limits_C \frac{ds}{y^2}$, where C is given by $y=\cosh x$, $x\in\mathbb{R}$,

   (g) $\int\limits_C z\,ds$, where C is a circle given by intersection of $x^2+y^2=z^2$ with $y^2=x$ from the point $O(0,0,0)$ to the point $A(1,1,\sqrt{2})$,

   (h) $\int\limits_C xy\,ds$, where C is the contour of the square $|x|+|y|=a$ $(a>0)$,

   (i) $\int\limits_C x\,ds$, where C is the arc of the logarithmic spiral $r=ae^{k\varphi}$, $(k>0)$, which lies inside the circle $r\le a$,

   (j) $\int\limits_C \frac{ds}{x^2+y^2+z^2}$, where C is the first turn of the screw-line $x=a\cos t$, $y=a\sin t$, $z=bt$.

   (answers: a) $1+\sqrt{2}$; b) $2\pi^2(1+2\pi^2)$; c) 8; d) $\frac{2}{3}\pi(3+4\pi^2)\sqrt{2}$; e) $\frac{2}{3}\pi$; f) π; g) $\frac{1}{256\sqrt{2}}(100\sqrt{38}-72-17\log\frac{25+4\sqrt{38}}{17})$; h) 0; i) $\frac{2ka^2\sqrt{1+k^2}}{1+4k^2}$; j) $\frac{\sqrt{a^2+b^2}}{ab}\arctan\frac{2\pi b}{a}$)

2. Find a mass of

   (a) the first turn of the screw-line $x=\cos t$, $y=\sin t$, $z=3t$, whose density at a point (x,y,z) equals $x^2+y^2+z^2$,

   (b) a curve $x=2\cos t$, $y=2\sin t$, $t\in[0,2\pi]$, whose density at a point (x,y) equals $|x|$,

   (c) a curve $x=at$, $y=\frac{1}{2}at^2$, $z=\frac{1}{3}at^3$, $a>0$, $t\in[0,1]$, whose density at a point (x,y,z) equals $\sqrt{\frac{2y}{a}}$,

   (d) an arc of a parabola $y^2=2px$ $(x\in[0,p/2])$, whose density at a point (x,y) equals $|y|$.

3. Calculate the following integrals:

   (a) $\int\limits_C (x^2 - 2xy)\,dx + (y^2 - 2xy)\,dy$, where C is the parabola $y = x^2$, $-1 \le x \le 1$,

   (b) $\int\limits_C (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$, where C is defined as $y = 1 - |x - 1|$, $0 \le x \le 2$,

   (c) $\oint\limits_C (x + y)\,dx - (x - y)\,dy$, where C is the circle $x^2 + y^2 = 2y + 1$ with counter-clockwise orientation,

   (d) $\oint\limits_C \dfrac{(x + y)\,dx - (x - y)\,dy}{x^2 + y^2}$, where C is the circle $x^2 + y^2 = 2y + 1$ with counter-clockwise orientation,

   (e) $\int\limits_C y^2\,dx + x^2\,dy$, where C is the upper half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ traced clockwise,

   (f) $\int\limits_C (y - z)\,dx + (z - x)\,dy + (x - y)\,dz$, where C is a turn of the screw-line $x = 2\cos t$, $y = 2\sin t$, $z = 3t$,

   (g) $\int\limits_C (y^2 - z^2)\,dx + 2yz\,dy - x^2\,dz$, where C is a curve $x = t$, $y = t^2$, $z = t^3$, $(0 \le t \le 1)$, which runs in the direction of increasing parameter t,

   (answer: $\frac{1}{35}$)

   (h) $\oint\limits_C (y^2 - z^2)\,dx + (z^2 - x^2)\,dy + (x^2 - y^2)\,dz$, where C is a boundary of the part of the sphere $x^2 + y^2 + z^2 = 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$, oriented so that the inner side of the surface is to the left of C.

   (answer: -4)

4. Find the potential function of a force $F = (X, Y, Z)$ and determine the work done by the force over a given path if:

   (a) $X = 0$, $Y = 0$, $Z = -mg$ (force of gravity) and the material point is moved from position $A(x_1, y_1, z_1)$ to position $B(x_2, y_2, z_2)$,

   (answer: potential: $U = mgz$, work: $mg(z_l - z_2)$)

   (b) $X = -\frac{\mu x}{r^3}$, $Y = -\frac{\mu y}{r^3}$, $Z = -\frac{\mu z}{r^3}$, where $\mu = \text{const}$ and $r = \sqrt{x^2 + y^2 + z^2}$ (Newton attractive force) and the material point moves from position $A(a, b, c)$ to infinity,

   (answer: potential: $U = \dfrac{\mu}{r}$, work: $\dfrac{\mu}{\sqrt{a^2 + b^2 + c^2}}$)

   (c) $X = -k^2 x$, $Y = -k^2 y$, $Z = -k^2 z$, where $k = \text{const}$ (elastic force), and the initial point of the path is located on the sphere $x^2 + y^2 + z^2 = R^2$, while the terminal point is located on the sphere $x^2 + y^2 + z^2 = r^2$, $R > r$.

   (answer: potential: $U = -\dfrac{k^2}{2}(x^2 + y^2 + z^2)$, work: $\dfrac{k^2}{2}(R^2 - r^2)$)

5. Show that the integrand functions are total differentials, find the antiderivative and compute integrals:

   (a) $\int\limits_{(0,1)}^{(2,3)} (x+y)\,dx + (x-y)\,dy$,

   (b) $\int\limits_{(0,1)}^{(3,-4)} x\,dx + y\,dy$,

   (c) $\int\limits_{(2,1)}^{(1,2)} \dfrac{y\,dx - x\,dy}{x^2}$, along a curve which does not intersect the y-axis,

   (d) $\int\limits_{(0,0)}^{(a,b)} e^x(\cos y\,dx + \sin y\,dy)$.

6. Show that the integrand functions are total differentials and compute integrals:

   (a) $\int\limits_{(1,1,1)}^{(2,3,-4)} x\,dx + y^2\,dy - z^3\,dz$,

   (b) $\int\limits_{(1,1,1)}^{(a,b,c)} yz\,dx + zx\,dy + xy\,dz$,

   (c) $\int\limits_{(0,0,0)}^{(3,4,5)} \dfrac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$,

   (d) $\int\limits_{(1,1,1)}^{(x,y,\frac{1}{xy})} \dfrac{yz\,dx + zx\,dy + xy\,dz}{xyz}$.

7. Show that the integrand functions are total differentials and find the antiderivative function U if:

   (a) $dU = (x^2+2xy-y^2)dx + (x^2-2xy-y^2)dy$,

   (b) $dU = \dfrac{y\,dx - x\,dy}{3x^2-2xy+3y^2}$,

   (c) $dU = (2x+3y)dx + (3x-4y)dy$,

   (d) $dU = e^{x-y}\big[(1+x+y)dx + (1-x-y)dy\big]$,

   (e) $dU = \dfrac{dx}{x+y} + \dfrac{dy}{x+y}$.

8. Show that the integrand functions are total differentials and find the antiderivative function U if:

   (a) $dU = \dfrac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$,

   (b) $dU = (x^2-2yz)\,dx + (y^2-2xz)\,dy + (z^2-2xy)\,dz$.

9. Using Green's theorem find the areas of figures bounded by the following curves:

   (a) the ellipse $x = a\cos t$, $y = b\sin t$,

   (b) the asteroid $x = a\cos^3 t$, $y = b\sin^3 t$,

   (c) the curve $(x+y)^2 = axy$,

   (d) the cardioid $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$.

10. Use Green's theorem to compute:

   (a) $\oint\limits_C xy^2 dx \, - x^2 y \, dy$, where C is a circle $x^2 + y^2 = a^2$,

   (b) $\oint\limits_C 2(x^2+y^2) dx \, + (x+y)^2 \, dy$, where C is the contour of the triangle (traced in the positive direction) with vertices at the points $A(1,1)$, $B(2,2)$, and $C(1,3)$; verify the result obtained by computing the integral directly,

   (c) $\oint\limits_C 2(x+y)^2 dx \, - (x^2+y^2) \, dy$, where C is the boundary of the triangle (traced in the negative direction) with vertices at the points $A(1,1)$, $B(3,2)$, and $C(2,5)$,

   (d) $\oint\limits_C 2xy^3 dx \, + 4x^2y^2 \, dy$, where C is the boundary of the region enclosed by $y = 0$, $x = 1$, $y = x^3$, traced in the positive direction,

   (e) $\oint\limits_C e^x((2-\cos y)\, dx - (\sin y - 2y)\, dy)$, where C is the contour bounding the region $0 \le x \le \pi$, $0 \le y \le \sin x$, oriented clockwise,

   (f) $\int\limits_{p(AO)} (e^x \sin y - y)\, dx + (e^x \cos y - 1)\, dy$, where $p(AO)$ is the upper semicircle $x^2 + y^2 = 4x$ from the point $A(4,0)$ to the point $O(0,0)$.
   (*Hint: Close the curve $p(AO)$ with the line segment AO.*)

Vector fields

11. Find out whether the given vector field $\vec{F}$ has a potential U, and find U if the potential exists:

   (a) $\vec{F} = (5x^2y - 4xy)\vec{i} + (3x^2 - 2y)\vec{j}$,

   (b) $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$,

   (c) $\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$.

12. Verity that a given vector field is potential and compute integrals:

   (a) $\int\limits_{(1,-1)}^{(1,1)} (x-y)(dx - dy)$,

   (b) $\int\limits_{(1,0)}^{(6,8)} \dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$,

(c) $\int\limits_{(1,2,3)}^{(6,1,1)} yz\,dx + xz\,dy + xy\,dz.$

13. Verify that the following fields are gradient fields. Find the potential U and draw the streamlines perpendicular to the equipotentials $U(x,y) = c$.

    (a) $\vec{F} = \vec{i} + 2\vec{j}$ (constant field),

    (b) $\vec{F} = x\vec{i} + \vec{j}$,

    (c) $\vec{F} = x\vec{i} + \vec{j}$,

    (d) $\vec{F} = \cos(x+y)\vec{i} + \cos(x+y)\vec{j}$,

    (e) $\vec{F} = x^2\vec{i} + y^2\vec{j}$,

    (f) $\vec{F} = \dfrac{1}{y}\vec{i} - \dfrac{x}{y^2}\vec{j}$.

14. Compute $\int \vec{F} \cdot d\vec{R}$ along the straight line $\vec{R} = t\vec{i} + t\vec{j}$ and the parabola $\vec{R} = t\vec{i} + t^2\vec{j}$ from $(0,0)$ to $(1,1)$. When $\vec{F}$ is a gradient field, use its potential $U(x,y)$.

    (a) $\vec{F} = \vec{i} - 2\vec{j}$,

    (b) $\vec{F} = 2xy^2\vec{i} + 2x^2y\vec{j}$,

    (c) $\vec{F} = y\vec{i} - x\vec{j}$,

    (d) $\vec{F} = x^2\vec{j}$.

15. Find the work in moving from $(1,0)$ to $(0,1)$. When $\vec{F}$ is conservative, construct its potential U. Choose your own path when $\vec{F}$ is not conservative.

    (a) $\vec{F} = \vec{i} + y\vec{j}$,

    (b) $\vec{F} = y\vec{i} + \vec{j}$,

    (c) $\vec{F} = e^y\vec{i} + xe^y\vec{j}$,

    (d) $\vec{F} = -y^2\vec{i} + x^2\vec{j}$.

Surface integrals of the first type (unoriented)

16. Evaluate the following surface integrals of the first type

    (a) $\iint\limits_S (x^2 + y^2)\,dS$, where S is the sphere $x^2 + y^2 + z^2 = a^2$;
    (answer: $\frac{8}{3}\pi a^4$)

    (b) $\iint\limits_S \sqrt{x^2 + y^2}\,dS$, where S is the lateral surface of the cone $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0$, $0 \le z \le b$;
    (answer: $2\pi a^2 \frac{\sqrt{a^2+b^2}}{3}$)

(c) $\displaystyle\iint\limits_S \frac{dS}{(1+x+y)^2}$, where S is the surface of the tetrahedron $x+y+z \le 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$;
(answer: $\frac{3-\sqrt{3}}{2} + (\sqrt{3}-1)\ln 2$)

(d) $\displaystyle\iint\limits_S z\,dS$, where S is the surface given as $x = u\cos v$, $y = u\sin v$, $z = v$, $u \in [0,a]$, $v \in [0, 2\pi]$.
(answer: $\pi^2\big(a\sqrt{1+a^2} + \ln(a+\sqrt{1+a^2})\big)$)

17. Find the mass of the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, if the surface density at the point $M(x,y,z)$ is equal to xyz.

    (answer: $\frac{3}{4}$)

18. Determine the coordinates of the centre of gravity of a homogeneous parabolic envelope $3z = x^2 + y^2$ $(0 \le z \le 3)$.

    (answer: $\frac{25\sqrt{5}+1}{10(5\sqrt{5}-1)}$)

19. Find the moment of inertia of a part of the lateral surface of the cone $z = \sqrt{x^2+y^2}$ $(0 \le z \le h)$ about the z-axis.

    (answer: $\frac{\pi^2\sqrt{2}}{2}h^4$)

20. Compute static moments of a part of the plane $x+y+z = a$, $x \ge 0$, $y \ge 0$, $z \ge 0$, with respect to planes xy, yz, zx.

    (answer: $\frac{a^3}{2\sqrt{3}}$)

21. Compute $\displaystyle\iint\limits_{x+y+z=t} f(x,y,z)\,dS$, where

$$f(x,y,z) = \begin{cases} 1 - x^2 - y^2 - z^2 & \text{if } x^2+y^2+z^2 \le 1, \\ 0 & \text{if } x^2+y^2+z^2 > 1. \end{cases}$$

    (answer: $\frac{\pi}{18}(3-t^2)^2$ for $|t| \le \sqrt{3}$; 0 for $|t| > \sqrt{3}$)

Surface integrals of the second type

22. Evaluate the following surface integrals of the second type

    (a) $\displaystyle\iint\limits_S yz\,dydz + xz\,dzdx + xy\,dxdy$, where S is the external side of the surface of a tetrahedron bounded by the planes $x = 0$, $z = 0$, $x+y+z = a$;

    (b) $\displaystyle\iint\limits_S z\,dxdy$, where S is the external side of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$;

(c) $\displaystyle\iint\limits_S z^2\,dx\,dy + x^2\,dy\,dz + y^2\,dz\,dx$, where S is the external side of the hemisphere $x^2 + y^2 + z^2 = a^2$ $(z \geq 0)$;

(answer: (a) 0; (b) $\frac{4}{3}\pi abc$; (c) $\frac{\pi a^4}{2}$)