IBM – Algebra Liniowa (ALL) – Zestaw 9 (powtórzeniowy)

1) i) $\det\begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} = 2a - a^3 = -a(a^2-2) = -a(a-\sqrt{2})(a+\sqrt{2}).$

minor $= 0\cdot a - 1\cdot 1 = -1 \neq 0 \Rightarrow \mathrm{rk}\begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} \geq 2.$

$$\therefore \quad \mathrm{rk}\begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} = \begin{cases} 2 & a \in \{\pm\sqrt{2}, 0\} \\ 3 & a \in \mathbb{R}\setminus\{\pm\sqrt{2}, 0\}. \end{cases}$$

ii) $\det\begin{bmatrix} 2 & 1 & 4 \\ 5 & 3 & 9 \\ 4 & 6 & a \end{bmatrix} = 6a + 36 + 120 - 48 - 5a - 108 = a$

minor $= 2\cdot 3 - 1\cdot 5 = 1 \neq 0 \Rightarrow \mathrm{rk}\begin{bmatrix} 2 & 1 & 4 \\ 5 & 3 & 9 \\ 4 & 6 & a \end{bmatrix} \geq 2.$

$$\therefore \quad \mathrm{rk}\begin{bmatrix} 2 & 1 & 4 \\ 5 & 3 & 9 \\ 4 & 6 & a \end{bmatrix} = \begin{cases} 2 & a = 0 \\ 3 & a \neq 0. \end{cases}$$

iii) $\mathrm{rk}\begin{bmatrix} 2 & 1 & a & 1 \\ 3 & 1 & 0 & 3 \\ 1 & 1 & 4 & -1 \end{bmatrix} \leq 3$, bo macierz ma 3 wiersze

minor $= 8 + 3a - a - 12 = 2a - 4 \Rightarrow \mathrm{rk} \geq 3$ dla $a \neq 2$.

• dla $a = 2$:

$$\mathrm{rk}\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 1 & 0 & 3 \\ 1 & 1 & 4 & -1 \end{bmatrix} \underset{+k_4\ +k_4\ +4k_4}{=} \mathrm{rk}\begin{bmatrix} 3 & 2 & 6 & 1 \\ 6 & 4 & 12 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \underset{:3\ :2\ :6}{=} \mathrm{rk}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = 2.$$

minor $= 3\cdot 1 - 1\cdot 2 = 1 \neq 0$

$$\therefore \quad \mathrm{rk}\begin{bmatrix} 2 & 1 & a & 1 \\ 3 & 1 & 0 & 3 \\ 1 & 1 & 4 & -1 \end{bmatrix} = \begin{cases} 2 & a = 2 \\ 3 & a \in \mathbb{R}\setminus\{2\}. \end{cases}$$

2)
$$\begin{cases} 3x + 2y - z = 1 \\ x + 3y - z = 0 \\ 2x - y - 3z = 3 \end{cases}$$

• metoda Cramera

$$W = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 3 & -1 \\ 2 & -1 & -3 \end{vmatrix} = -21 \qquad W_y = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 3 & -3 \end{vmatrix} = 7$$

$$W_x = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 3 & -1 & -3 \end{vmatrix} = -7 \qquad W_z = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 14$$

$$\begin{cases} x = \frac{W_x}{W} = \frac{1}{3} \\ y = \frac{W_y}{W} = -\frac{1}{3} \\ z = \frac{W_z}{W} = -\frac{2}{3}. \end{cases}$$

• metodą macierzy odwrotnej

$$\underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 1 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

.. A^{-1} metodą dopełnień algebraicznych

$$\det A = W = -21$$

$$A^D = \begin{bmatrix} \begin{vmatrix} 3 & -1 \\ -1 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -10 & 1 & -7 \\ 7 & -7 & 7 \\ 1 & 2 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \left(A^D\right)^T = \frac{1}{-21} \begin{bmatrix} -10 & 7 & 1 \\ 1 & -7 & 2 \\ -7 & 7 & 7 \end{bmatrix} = \begin{bmatrix} \frac{10}{21} & -\frac{1}{3} & -\frac{1}{21} \\ -\frac{1}{21} & \frac{1}{3} & -\frac{2}{21} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

.. A^{-1} metodą Gaussa

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 2 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \updownarrow \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ 3 & 2 & -1 & 1 & 0 & 0 \\ 2 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ -3w_1 \\ -2w_1 \end{matrix} \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ 0 & -7 & 2 & 1 & -3 & 0 \\ 0 & -7 & -1 & 0 & -2 & 1 \end{array}\right] \begin{matrix} -w_3 \\ +2w_3 \\ \ \end{matrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 10 & 0 & 0 & 3 & -1 \\ 0 & -21 & 0 & 1 & -7 & 2 \\ 0 & -7 & -1 & 0 & -2 & 1 \end{array}\right] \begin{matrix} \\ :(-21) \\ :(-1) \end{matrix}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 10 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & -\frac{1}{21} & \frac{1}{3} & -\frac{2}{21} \\ 0 & 7 & 1 & 0 & 2 & -1 \end{array}\right] \begin{matrix} -10w_2 \\ \ \\ -7w_2 \end{matrix} \rightarrow \left[\begin{array}{c|ccc} & \frac{10}{21} & -\frac{1}{3} & -\frac{1}{21} \\ I & -\frac{1}{21} & \frac{1}{3} & -\frac{2}{21} \\ & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] = \left[I \,|\, A^{-1}\right]$$

$$\therefore A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{10}{21} \cdot 1 - \frac{1}{3} \cdot 0 - \frac{1}{21} \cdot 3 \\ -\frac{1}{21} \cdot 1 + \frac{1}{3} \cdot 0 - \frac{2}{21} \cdot 3 \\ \frac{1}{3} \cdot 1 - \frac{1}{3} \cdot 0 - \frac{1}{3} \cdot 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

• metodą eliminacji (Gaussa):

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & -1 & -3 & 3 \end{array}\right] \updownarrow \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 3 & 2 & -1 & 1 \\ 2 & -1 & -3 & 3 \end{array}\right] \begin{matrix} \\ -3w_1 \\ -2w_1 \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -7 & 2 & 1 \\ 0 & -7 & -1 & 3 \end{array}\right] \begin{matrix} \\ \\ -w_2 \end{matrix} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -7 & 2 & 1 \\ 0 & 0 & -3 & 2 \end{array}\right] \begin{matrix} \\ \\ :(-3) \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -7 & 2 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right] \begin{matrix} +w_2 \\ -2w_2 \\ \ \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -\frac{2}{3} \\ 0 & -7 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right] \begin{matrix} \\ :(-7) \\ \ \end{matrix} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \xrightarrow{-3w_2} \left[\begin{array}{c|c} I & \begin{matrix} 1/3 \\ -1/3 \\ -2/3 \end{matrix} \end{array}\right] \Rightarrow \begin{cases} x = 1/3 \\ y = -1/3 \\ z = -2/3. \end{cases}$$

3) Niech $A = \begin{bmatrix} 1 & a & -1 \\ \boxed{\begin{matrix} 1 & 1 \\ 1 & 3a \end{matrix}} & \begin{matrix} -a \\ 3 \end{matrix} \end{bmatrix}$; $A_r = \begin{bmatrix} 1 & a & -1 & 5 \\ 1 & 1 & -a & 2a \\ 1 & 3a & 3 & 9 \end{bmatrix}$

$\operatorname{rk} A = ?$, $\operatorname{rk} A_r = ?$ Zawsze $\operatorname{rk} A \leq \operatorname{rk} A_r$

$\operatorname{rk} A$: $\det A = 3 - a^2 - 3a + 1 - 3a + 3a^2 = 2a^2 - 6a + 4 =$

$= 2(a^2 - 3a + 2) = 2(a-2)(a-1).$

minor $= 3a - 1 \neq 0$ dla $a \neq \frac{1}{3}$.

Zatem $\operatorname{rk} A = \begin{cases} 2 & a \in \{1, 2\} \\ 3 & a \in \mathbb{R} \setminus \{1, 2\}. \end{cases}$

- gdy $a = 1$:

$$\operatorname{rk} A_r = \operatorname{rk} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 1 & 1 & -1 & 2 \\ 1 & 3 & 3 & 9 \end{bmatrix} \begin{matrix} \\ -w_1 \\ -w_1 \end{matrix} = \operatorname{rk} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 0 & 0 & -3 \\ 0 & 2 & 4 & 4 \end{bmatrix} = 3.$$

- gdy $a = 2$

$$\operatorname{rk} A_r = \operatorname{rk} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 1 & 1 & -2 & 4 \\ 1 & 6 & 3 & 9 \end{bmatrix} \begin{matrix} \\ -w_1 \\ -w_1 \end{matrix} = \operatorname{rk} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & -1 & -1 & -1 \\ \cancel{0} & \cancel{4} & \cancel{4} & \cancel{4} \end{bmatrix} = 2$$

Zatem:

- dla $a = 2$: układ ma ∞-wiele rozwiązań zależnych od $3-2=1$ parametru
- dla $a = 1$: układ jest sprzeczny (ma 0 rozwiązań)
- dla $a \in \mathbb{R} \setminus \{1, 2\}$: układ ma dokładnie jedno rozwiązanie.

4) Niech $A = \begin{bmatrix} 1 & a \\ a & 1 \\ a & a \end{bmatrix}$, $A_r = \begin{bmatrix} 1 & a & a^2 - a \\ a & 1 & 2a \\ a & a & a^2 \end{bmatrix}$

Wtedy $\operatorname{rk} A \leq 2$, $\operatorname{rk} A_r \leq 3$. ; $\operatorname{rk} A \leq \operatorname{rk} A_r$.

$\det A_r = \underline{a^2} + \underline{2a^3} + \underline{a^4} - \underline{a^3} - \underline{a^3} + \underline{a^2} - \underline{a^4} - \underline{2a^2} = 0$

Zatem $\operatorname{rk} A_r \leq 2$.

minor $= 1 - a^2 = (1-a)(1+a)$

minor $= a^2 - a = a(a-1)$.

$\Big\}$ dla $a \neq 1$ $\operatorname{rk} A = 2$.

dla $a = 1$: $\operatorname{rk} A = \operatorname{rk} \begin{bmatrix} 1 & 1 \\ \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} \end{bmatrix} = 1$

$$\operatorname{rk} A_r = \begin{bmatrix} 1 & \boxed{\begin{matrix} 1 & 0 \\ 1 & 2 \end{matrix}} \\ 1 & \\ 1 & \begin{matrix} 1 & 1 \end{matrix} \end{bmatrix} = 2.$$

minor $= 1\cdot 2 - 0\cdot 1 = 2 \neq 0.$

Zatem:

- dla $a=1$ $\quad \operatorname{rk} A = 1 < 2 = \operatorname{rk} A_r \Rightarrow$ układ jest sprzeczny
- dla $a \neq 1$ $\quad \operatorname{rk} A = 2 = \operatorname{rk} A_r \Rightarrow$ układ ma jedyne rozwiązanie.

5)

$L = (A+B)^2 = (A+B)(A+B) = A^2 + A\cdot B + B\cdot A + B^2$

$P = A^2 + 2AB + B^2$

$$L = P \iff A^2 + AB + BA + B^2 = A^2 + 2AB + B^2 \iff BA = AB.$$

Niech $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Wtedy

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3a-b & a \\ 3c-d & c \end{bmatrix}; \quad BA = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a+c & 3b+d \\ -a & -d \end{bmatrix}$$

$$AB = CD \iff \begin{cases} (1)\ 3a + c = 3a - b & \Rightarrow b + c = 0 \\ (2)\ a = 3b + d & \\ (3)\ 3c - d = -a & \\ (4)\ c = -d & \Rightarrow d + c = 0 \end{cases} \qquad b = d \quad (5)$$

z (2) i (5): $a = 4b$

z (3) i (5): $4c = -a$ $\qquad \therefore A = \begin{bmatrix} 4b & b \\ -b & b \end{bmatrix}$

z (1): $-b = c$

Ale

$(A + e\cdot I)B = AB + eB$

$B(A + eI) = BA + eB$

Zatem jeśli $AB = BA$, to $(A+eI)B = B(A+eI)$

W takim razie $AB = BA \iff A = \begin{bmatrix} 4b+e & b \\ -b & b+e \end{bmatrix}$, $b, e \in \mathbb{R}$.