2 IN JUNE

1 IN SEPTEMBER

TUTORIALS 0 - 40

EXAM 0 - 60

0 - 100

2 TESTS

+ ACTIVITY

(30)

IF TUTORIALS > 32

NECESSARY CONDITION

EXAM > 30

91 - 100	5
81 - 90	4.5
71 - 80	4
61 - 70	3,5
$\boxed{51}$ - 60	3

$$\boxed{y'(x) = y(x)} \quad , \quad y'(x) = x \cdot (y(x))^2 \quad , \quad \boxed{y''(x) = y(x)}$$

INDEPENDENT VARIABLE

DEP. VARIABLE

$$\boxed{y(x) = Ce^x} \qquad y(x) = De^{-x}$$

$$y'(x) = -De^{-x}$$

$$\Downarrow$$

$$y''(x) = De^{-x}$$

$$y(x) = e^x \quad [= \exp(x)]$$

$$y(x) = 2e^x$$

$$\boxed{y(x) = Ce^x}$$

$$\boxed{y(x) = Ce^x + De^{-x}}$$

$C, D \in \mathbb{R}$ CONSTANTS

$$\Downarrow$$

$$y'(x) = Ce^x = y(x) \Rightarrow y''(x) = Ce^x = y(x)$$

$y'(x) = 0 \iff y(x) = C$

$y'(x) = x \iff y'(x) = \left(\frac{x^2}{2}\right)'$

$\Updownarrow$

$\left(y(x) - \frac{x^2}{2}\right)' = 0$

$\Updownarrow$

$y(x) - \frac{x^2}{2} = C$

$\Updownarrow$

$y(x) = \frac{x^2}{2} + C$

$\int x^a dx = \frac{x^{a+1}}{a+1} + C$

$a \neq -1$

$a = -\frac{1}{2}$

$\int \frac{1}{\sqrt{x}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= 2\sqrt{x} + C$

$\int x \, dx = \frac{x^2}{2} + C$

$(y')' = y''(x) = x$

$\Updownarrow$

$y' = \int x \, dx = \frac{x^2}{2} + C \Rightarrow y(x) = \int \left(\frac{x^2}{2} + C\right) dx =$

$= \frac{x^3}{6} + Cx + D$

$\int x^2 dx = \frac{x^3}{3} + C$

$\int C \, dx = Cx + D$

$\cancel{C}$

$\frac{x^3}{6} + \cancel{Cx + C}$

$$\boxed{y'(x) = y(x)} \quad \Leftarrow \quad y(x) = e^x$$

METHOD 1

DEFINE $z(x) = \dfrac{y(x)}{e^x}$ $\left[\dfrac{e^x}{e^x} = 1\right]$

$$z'(x) = \frac{\overset{y(x)}{\overbrace{y'(x)}} \cdot e^x - y(x) \cdot e^x}{(e^x)^2} = \frac{0}{(e^x)^2} = 0$$

(NEW UNKNOWN)

$C \in \mathbb{R}$

$$z'(x) = 0 \Rightarrow z(x) = C \Rightarrow y(x) = z(x) \cdot e^x = \boxed{Ce^x}$$

METHOD 2

$$u(x) = \ln y(x)$$

IF $y(x) > 0$, $w(x) = \ln(-y(x))$ IF $y(x) < 0$

$\ln e^x = x$

$$(\ln(-y(x)))' = \frac{-y'(x)}{-y(x)} = 1$$

$$u'(x) = \frac{1}{y(x)} \cdot y'(x) = \frac{y(x)}{y(x)} = 1 \Rightarrow w(x) = \int 1 \, dx = x + C$$

ALSO A SOLUTION

$$y(x) = 0$$

$$u'(x) = 1$$

$$\Downarrow$$

$D = \pm e^C$

$$y(x) = \pm e^{x+C} = e^x e^C = De^x$$

'METHOD' 3 $\boxed{y'(x) = y(x)}$ $w(x) = y(x) - e^x$ $y' = y$

$$w'(x) = (y(x) - e^x)' = y'(x) - e^x \overset{y'=y}{=} y(x) - e^x = w(x)$$

$$\boxed{w'(x) = w(x)}$$

NO PROGRESS

SOLVE $\boxed{y''(x) = y(x)}$

$\underline{y(x) = e^x}$ IS $\underline{\text{ONE}}$ OF SOLUTIONS

SUBSTITUTE $z(x) = \dfrac{y(x)}{e^x}$

$$y(x) = z(x)\, e^x$$

$$y'(x) = z'(x)\, e^x + z(x)\, e^x$$

$$y''(x) = \underbrace{z''(x)\, e^x + z'(x)\, e^x}_{} + \underbrace{z'(x)\, e^x + z(x)\, e^x}_{}$$

$$\cancel{y(x)} = z(x)\, \cancel{e^x}$$

$$y''(x) = (z''(x) + 2z'(x) + \cancel{z(x)})\, \cancel{e^x}$$

$$\boxed{z''(x) + 2z'(x) = 0}$$

$$\boxed{y''(x) = y(x)}$$

$$(e^{ax})' = ae^{ax}$$

$$z''(x) + 2z'(x) = 0$$

SUBSTITUTE $u(x) \overset{\circledast}{=} z'(x)$

$$\boxed{u'(x) + 2u(x) = 0} \cdot e^{2x}$$

$$u'(x) = \underbrace{-2}_{"a"} u(x)$$

$$u'(x) \cdot e^{2x} + u(x) \cdot 2e^{2x} = 0$$

$$\parallel$$

$$(e^{-x})' = -e^{-x} \qquad \underline{(u(x)e^{2x})'}$$

$$\int 0 \, dx = C$$

$$u(x) = Ce^{-2x}$$

$$(e^{-x})'' = -(-e^{-x}) = e^{-x} \qquad u(x) \cdot e^{2x} = C$$

$$\Downarrow$$

$$u(x) = Ce^{-2x}$$

$$\circledast \quad z'(x) = Ce^{-2x}$$

$$\Downarrow$$

$$z(x) = \int Ce^{-2x} dx = -\frac{C}{2} e^{-2x} + D$$

$$y(x) = z(x) \cdot e^{x} = \left(-\frac{C}{2} e^{-2x} + D\right) e^{x} = De^{x} - \frac{C}{2} e^{-x} = \boxed{C_1 e^{x} + C_2 e^{-x}}$$

$$y'(x) = y(x) \qquad y'(x) = a\,y(x) \qquad y'(x) = x \cdot y(x)$$

$$\Updownarrow \qquad \Updownarrow \qquad \Downarrow$$

$$y(x) = Ce^{x} \qquad y(x) = C e^{ax} \qquad y(x) = ?$$

$$\left(\ln|y(x)|\right)' = \frac{y'(x)}{y(x)} = 1 \text{ OR } a \text{ OR } x \quad \Big| \cdot \int \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\ln|y(x)| = x + C \text{ OR } ax + C \text{ OR } \frac{x^2}{2} + C \;= \int x\,dx$$

- $\ln|y(x)| = x + C \Rightarrow y(x) = \pm e^{x+C} = \pm e^{C} e^{x} = D e^{x}$
- $\ln|y(x)| = ax + C \Rightarrow y(x) = \pm e^{ax+C} = \pm e^{C} e^{ax} = D e^{ax}$
- $\ln|y(x)| = \frac{x^2}{2} + C \Rightarrow y(x) = \pm e^{\frac{x^2}{2}+C} = \pm e^{C} e^{\frac{x^2}{2}} = D e^{\frac{x^2}{2}}$

$D = \pm e^{C}$

SEPARATION OF VARIABLES

SEPARABLE DIFF. EQUATION:

$$y'(x) = f(x)\cdot g(y)$$

$$\frac{y'(x)}{g(y(x))} = f(x)$$

$$\Downarrow$$

$$\int \frac{y'(x)}{g(y(x))}\,dx = \int f(x)\,dx$$

$dy =$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

$$\boxed{y' = (x+y)\,y}$$

NOT SEPARABLE

$$(\ln|y(x)|)' = \frac{y'(x)}{y(x)} = x + y(x)$$

$$\ln|y(x)| = \int (x + y(x))\,dx =$$

$$= \frac{x^2}{2} + \underbrace{\int y(x)\,dx}_{=\,?}$$

$$\cancel{\neq\; y\cdot x + C}$$

$\boxed{\frac{dy}{dx} =} \; y' = x^2 \cdot y^2$ $\quad \cancel{y' = x^2 + y^2} \quad y'(x) = x^2 [y(x)]^2 \qquad x' + y' = xy$?

$-\left(\frac{1}{y}\right)' = \frac{y'}{y^2} = x^2 \qquad\qquad \frac{dy}{dx} = x^2 y^2$

SEPARATION $\longrightarrow$

$$-\frac{1}{y} = \int x^2\,dx = \frac{x^3}{3} + C \qquad\qquad \int \frac{dy}{y^2} = \int x^2\,dx$$

$$y = -\frac{1}{\frac{x^3}{3} + C} = \frac{3}{-3C - x^3} = \frac{3}{D - x^3} \quad \longleftarrow \quad -\frac{1}{y} = \frac{x^3}{3} + C$$

WE CAN CHECK: $y' = 3\left(-\frac{1}{(D - x^3)^2}\right) \cdot (-3x^2) = x^2 \left(\frac{3}{D - x^3}\right)^2 = x^2 \cdot y^2$ ✓

$$y' = \underbrace{(y+x)^2}$$ $= y^2 + 2xy + x^2$

SUBSTITUTE $\underline{u(x) = y(x) + x}$ $[u = y + x]$

$$y(x) = u(x) - x$$

$$y'(x) = u'(x) - 1$$

$$u' - 1 = u^2$$

$$\frac{du}{dx} = u^2 + 1$$

$$\int \frac{du}{u^2+1} = \int dx$$

tg

$[\tan^{-1} u =]$ $\arctan u = x + C \Rightarrow u = \tan(x+C) \Rightarrow y(x) = \tan(x+C) - x$

GENERALLY $\underline{y' = f(ax+by+c)}$ $\leftarrow$ $u = ax+by+c$

$$y' = f\left(\frac{y}{x}\right) \longleftarrow u(x) = \frac{y(x)}{x}$$

$y(x) = u(x)\cdot x$

$y'(x) = u'(x)\cdot x + u(x)\cdot 1$

EX. SET 3; 1a) $(x+y)\,y' + y = 0$

$$u'x + u = y' = \frac{-y}{x+y} = -\frac{y/x}{1+y/x} \qquad y/x = u(x)$$

$$u'x + u = -\frac{u}{1+u}$$

$$\frac{du}{dx}\cdot x = u'\cdot x = -\frac{u}{1+u} - u = -\frac{u^2+2u}{u+1}$$

$$\frac{1}{2}\int \frac{2u+2}{u^2+2u}\,du = -\int\frac{dx}{x} \qquad \frac{1}{2}\ln|u^2+2u| = -\ln|x| + \ln C \quad \Big|\, e^{\cdots}$$

$\int \frac{dx}{x} = \ln|x| + C$; $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$\sqrt{w^2+2w} = \frac{C}{x}$ $\qquad w = \frac{y}{x}$ $\qquad e^{\frac{1}{2}\ln x} = \sqrt{x}$

$$\sqrt{\frac{y^2+2yx}{x^2}} = \frac{C}{x}$$

$$\sqrt{y^2+2xy} = C$$

$$y^2+2xy\;[=C^2] = D \;\Rightarrow\; \boxed{x = \frac{D-y^2}{2y}}$$

ALSO 100% CORRECT

$$(y+x)^2 = y^2+2xy+x^2 = D+x^2$$

$$y+x = \pm\sqrt{D+x^2}$$

$$\boxed{y = -x \pm \sqrt{x^2+D}}$$ ← BEST

$y = x^2 \qquad \frac{dy}{dx} = \boxed{2x}$

$x = \sqrt{y} \qquad \frac{dx}{dy} = \boxed{\frac{1}{2x}} = \frac{1}{2\sqrt{y}}$

LINEAR DIFFERENTIAL EQUATIONS

1^{ST} ORDER $\quad a(x)\cdot y'(x) + b(x)\cdot y(x) = c(x)$; a, b, c — GIVEN FUNCTIONS

2^{ND} ORDER $\quad a(x)\cdot y''(x) + b(x)\cdot y'(x) + c(x)y(x) = d(x)$

$\vdots$

$$\sin x \cdot y' + \ln x \cdot y = e^x$$

(LINEAR WITH RESPECT TO y,

NO $\sin y, e^y, \frac{1}{y}, \ln y, y^2$

$(\sin x)\cdot y' =$

$\ne$

$\cancel{\sin(x\,y)}$

Ex. 0

$$xy' + y = x^2$$

$$\underbrace{xy' + y}_{=(x\cdot y)'}$$

$$\Rightarrow x\cdot y = \int x^2\,dx = \frac{x^3}{3} + C \quad \Big|:x$$

$$\underbrace{y = \frac{x^2}{3} + \boxed{\frac{C}{x}}}$$

Ex. $\frac{1}{2}$

$$xy' + 2y = x^2 \quad \Big|\cdot x$$

$$\underbrace{x^2 y' + 2x\cdot y}_{=(x^2\cdot y)'} = x^3$$

$$\Rightarrow x^2\cdot y = \int x^3\,dx = \frac{x^4}{4} + C \quad \Big|:x^2$$

$$\underbrace{y = \frac{x^2}{4} + \boxed{\frac{C}{x^2}}}$$

- $\underbrace{xy' + 2y}_{} = \underbrace{x^2}_{}$ $\qquad y(x) = \underbrace{\frac{C}{x^2}}_{} + \underbrace{\frac{x^2}{4}}_{}$

- $\underbrace{xy' + 2y}_{} = \underbrace{x^3}_{} \quad | \cdot x$

$$x^2 y' + 2xy = x^4$$

$$\overset{\shortparallel}{(x^2 \cdot y)'} \quad \Big| \Rightarrow x^2 \cdot y = \frac{x^5}{5} + C \Rightarrow y(x) = \underbrace{\frac{C}{x^2}}_{} + \underbrace{\frac{x^3}{5}}_{}$$

$$x(Cy)' + 2(Cy) = 0$$

HOMOGENEOUS EQUATION

$$\underbrace{xy' + 2y}_{} = \boxed{0}$$

$$y = \pm e^C x^{-2} = D x^{-2} = \frac{D}{x^2}$$

$$x \frac{dy}{dx} = -2y$$

$\exp(x) = e^x$

$$\frac{dy}{y} = -2\frac{dx}{x} \Rightarrow \overbrace{\ln|y| = -2\ln|x| + C}^{} \Rightarrow |y| = |x|^{-2} \cdot e^C$$

SOLVE: $\boxed{x y' + 2y = e^x}$ (*)

(METHOD OF VARIATION OF CONSTANTS)

STEP 1 CONSIDER AND SOLVE THE CORRESPONDING HOMOGENEOUS EQ. (HE)

$$x y' + 2y = 0$$

$$x \frac{dy}{dx} + 2y = 0 \Rightarrow \frac{dy}{y} = -2\frac{dx}{x} \Rightarrow y(x) = \frac{C}{x^2}$$

STEP 2 SUBSTITUTE $C(x) = y(x) \cdot x^2$; $y(x) = \frac{C(x)}{x^2}$

$$y'(x) = \frac{C'(x) \cdot x^2 - C(x) \cdot 2x}{(x^2)^2} = \frac{C'(x)}{x^2} - \frac{2C(x)}{x^3}$$

(*) BECOMES NOW

$$x\left(\frac{C'(x)}{x^2} - \frac{2C(x)}{x^3}\right) + 2\frac{C(x)}{x^2} = e^x$$

$$\frac{C'(x)}{x} - \cancel{\frac{2C(x)}{x^2}} + \cancel{\frac{2C(x)}{x^2}} = e^x \Rightarrow C'(x) = x e^x$$

$$C(x) = \int x e^x dx = x e^x - e^x + C$$

$$y(x) = \frac{x e^x - e^x + C}{x^2} = \frac{C}{x^2} + e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$\boxed{2b}$ $2xy' = y + \frac{3}{2}x^2 \iff 2xy' - y = \frac{3}{2}x^2$

$\boxed{HE}$: $2xy' = y$ $\cancel{2xy' = 0}$ $2xy' - y = 0$

$$2x\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \frac{1}{2}\int \frac{dx}{x} \Rightarrow \ln|y| = \frac{1}{2}\ln|x| + \ln C \Rightarrow y = \pm x^{1/2} \cdot C = \boxed{D\sqrt{x}}$$

GENERAL SOL. OF HE

LG : $y(x) = D(x)\sqrt{x}$; $y'(x) = D'(x)\sqrt{x} + D(x)\cdot\frac{1}{2\sqrt{x}}$

$$2x\left(D'\sqrt{x} + D\cdot\frac{1}{2\sqrt{x}}\right) = D\sqrt{x} + \frac{3}{2}x^2$$

$$2x^{3/2}D' + \cancel{D\sqrt{x}} = \cancel{D\sqrt{x}} + \frac{3}{2}x^2$$

$$D' = \frac{\frac{3}{2}x^2}{2x^{3/2}} = \frac{3}{4}x^{1/2} \Rightarrow D(x) = \frac{3}{4}\int x^{1/2}\,dx = \frac{3}{4}\cdot\frac{x^{3/2}}{3/2} + C = \frac{1}{2}x^{3/2} + C$$

FINALLY $y(x) = D(x)\sqrt{x}$ $\boxed{= \frac{1}{2}x^2 + C\sqrt{x}}$

3c $\quad xy' - y = y^{2} \quad \Big| : y^2$ (NOT LINEAR, BERNOULLI EQ.)

$n = 2$

$$\left[\; x\frac{y'}{y^2} - \frac{1}{y} = 1 \right. \qquad \left(\frac{1}{y}\right)' = -\frac{y'}{y^2} \qquad u = y^{1-n}$$

SUBST. $\boxed{w(x) = \frac{1}{y(x)}}$ $\quad y = \frac{1}{w}$

$(*)\quad -xw' - w = 1 \quad$ (LINEAR!)

sol. OF HE

HE $\quad -xw' - w = 0$

$$x\frac{dw}{dx} = -w \Rightarrow \int \frac{dw}{w} = -\int\frac{dx}{x} \Rightarrow \ln|w| = -\ln|x| + D \Rightarrow \boxed{w = \frac{C}{x}}$$

LET $\quad w(x) = \frac{C(x)}{x}\ ; \quad w'(x) = \frac{C'(x)}{x} - \frac{C(x)}{x^2}$

$$(*)\quad -x\left(\frac{C'(x)}{x} - \frac{C(x)}{x^2}\right) - \frac{C(x)}{x} = 1 \qquad -C'(x) = 1 \Rightarrow C(x) = -x + C$$

$$-C'(x) + \frac{C(x)}{\cancel{x}} - \frac{C(x)}{\cancel{x}} \qquad w(x) = \frac{-x+C}{x} \Rightarrow \boxed{y(x) = \frac{x}{C-x}}$$

1d) $(x^2+2xy)y' = y^2$

$$y' = \frac{y^2}{x^2+2xy} = \frac{\left(\frac{y}{x}\right)^2}{1+2\left(\frac{y}{x}\right)}$$

$$u = \frac{y}{x} \Leftrightarrow y = ux$$

$$y' = u'x + u$$

$$u'x + u = \frac{u^2}{2u+1}$$

$$\frac{du}{dx} \cdot x = \frac{u^2}{2u+1} - u = \frac{u^2 - 2u^2 - u}{2u+1} = \frac{-u^2 - u}{2u+1}$$

$$\int \frac{2u+1}{u^2+u}\, du = -\int \frac{dx}{x}$$

$$y = \frac{-x \pm \sqrt{x^2+4cx}}{2}$$

$$x = \frac{y^2}{c-y}$$

$$\ln|u^2+u| = -\ln|x| + \ln C \Rightarrow u^2+u = \frac{C}{x}$$

$$\left(\frac{y}{x}\right)^2 + \frac{y}{x} = \frac{C}{x} \quad \Big| \cdot x^2$$

$$\boxed{y^2 + xy = Cx}$$

BERNOULLI DIFFERENTIAL EQUATION

$$a(x) \cdot y'(x) + b(x) \cdot y(x) = c(x) \, [y(x)]^n \quad \Big| \div y^n \qquad n \in \mathbb{R} \setminus \{0,1\}$$

$$a(x) \boxed{\frac{y'(x)}{[y(x)]^n}} + b(x) \frac{1}{[y(x)]^{n-1}} = c(x)$$

SUBST: $w(x) = \frac{1}{[y(x)]^{n-1}}$

$\underbrace{w = y^{1-n}}$

$w' = (1-n) \cdot \boxed{y^{-n} \cdot y'}$

$$a(x) \boxed{\frac{1}{1-n} \cdot w'(x)} + b(x) \, w(x) = c(x)$$

LINEAR EQUATION FOR w

AFTER FINDING $w(x)$ WE HAVE

$$\underbrace{y(x) = w(x)^{\frac{1}{1-n}}}$$

$$(x^p)' = p \cdot x^{p-1}$$