$y''_{(x)} - 2y'_{(x)} - 3y_{(x)} = f(x) \quad (*)$

$y'' - 2y' - 3y = 0 \quad (HE)$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1)$

$r_1 = \boxed{3}; \; r_2 = \boxed{-1}$

GSHE

$y(x) = C_1 e^{3x} + C_2 e^{-x}$

$f(x)$	PREDICTED $y(x)$ (A SOLUTION OF $(*)$)
$x^2 + 1$	$Ax^2 + Bx + C$
$x e^x$	$(Ax+B)e^x$
$x e^{3x}$	$\boxed{x}(Ax+B)e^{3x}$
$(x^2-1)e^{-x}$	$\boxed{x}(Ax^2+Bx+C)e^{-x}$
$\sin 2x$	$A\sin 2x + B\cos 2x$
$x^2 \sin 3x$	$(Ax^2+Bx+C)\sin 3x + (Dx^2+Ex+F)\cos 3x$
$(x+1)e^{3x}\underbrace{\cos x}_{\frac{e^{ix}+e^{-ix}}{2}}$	$(Ax+B)e^{3x}\cos x + (Cx+D)e^{3x}\sin x$
$e^{(3+i)x}$	$3 \pm i \notin \{3, -1\}$

$\cancel{Ax^2 + C}$

$y(x) = Ax^2 + Bx + C$

$y'(x) = 2Ax + B$

$y''(x) = 2A$

$y'' - 2y' - 3y = 2A - 4Ax - 2B - 3Ax^2 - 3Bx - 3C$

$= (-3A)x^2 + (-4A - 3B)x + (2A - 2B - 3C)$

$= 1 \; x^2 + 0 \; x + 1$

$$\begin{cases} -3A = 1 \\ -4A - 3B = 0 \\ 2A - 2B - 3C = 1 \end{cases} \quad \begin{array}{l} A = \dots \\ B = \dots \\ C = \dots \end{array}$$

NO SYSTEM

NO PREDICTION METHOD

$y'' - 2y' - 3y = \frac{e^x}{x}$

$y'' - 2y' - 3y = \sin(x^2)$

Ex. (1) $y'' - 2y' + y = \frac{x^2+4}{e^x} = g_1(x)$

(2) $y'' - 2y' + y = \frac{e^x}{x^2+4} = g_2(x)$

$y'' - 2y' + y = 0$ (HG)

$r^2 - 2r + 1 = 0$

$(r-1)^2$ $r_1 = r_2 = 1$

G.Sol HE: $y(x) = C_1 e^x + C_2 x e^x$

(1) PREDICTION METHOD:

$-1 \neq 1$

$g_1(x) = (x^2+4) \cdot e^{-x}$

WE PREDICT

$y(x) = (Ax^2 + Bx + C) \cdot e^{-x}$

$y'(x) = (2Ax + B) e^{-x} - (Ax^2 + Bx + C) e^{-x} =$

$= (-Ax^2 + (2A - B)x + (B - C)) e^{-x}$

$y''(x) = 2Ae^{-x} - \boxed{2}(2Ax + B) e^{-x} + (Ax^2 + Bx + C) e^{-x} =$

$= (Ax^2 + (B - 4A)x + (2A - 2B + C)) e^{-x}$

$y'' - 2y' + y = e^{-x}(x^2+4)$

$(\text{something})\, e^{-x} = e^{-x}(x^2+4)$

$$y'' - 2y' + y = \frac{e^x}{x^2+4}$$

$$y(x) = z(x)e^x$$

$$z(x) = \frac{y(x)}{e^x} = y(x)e^{-x}$$

$$(z''+2z'+z)\cancel{e^x} - 2(z'+z)\cancel{e^x} + z\cancel{e^x} = \frac{\cancel{e^x}}{x^2+4} \quad \Big| : e^x$$

$$y' = z'e^x + ze^x = (z'+z)e^x$$

$$z'' + \cancel{2z'} + \cancel{z} - \cancel{2z'} - \cancel{2z} + \cancel{z} = \frac{1}{x^2+4}$$

$$y'' = [(z'+z)e^x]' =$$

$$z'' = \frac{1}{x^2+4} \Rightarrow z' = \frac{1}{2}\arctan\frac{x}{2} + C$$

$$= (z'+z)'e^x + (z'+z)e^x =$$

$$z = \frac{1}{2}\int \arctan\frac{x}{2}\,dx + C_x = (z''+2z'+z)e^x$$

$$y'' - 2y' + y = (x^2+4)e^{-x} \qquad y(x) = z(x)e^x$$

$$z' = \int (x^2+4)e^{-2x}\,dx$$

$$\ldots$$

$$= \ldots$$

$$e^x z'' = (x^2+4)e^{-x} \Rightarrow z'' = (x^2+4)e^{-2x}$$

$$z = \ldots$$

(1) $y'' - 2y' + y = f(x) = \frac{e^x}{x^2+4}$

(HE) $y'' - 2y' + y = 0 \longrightarrow r^2 - 2r + 1 = 0$

$(r-1)^2$

$c_1 e^{1x} + c_2 \boxed{x} e^{1x}$

$y(x) = C_1 y_1(x) + C_2 y_2(x) \leftarrow$ GSHE

(2) $y(x) = C_1(x) y_1(x) + C_2(x) y_2(x)$

$$y(x) = \left(-\frac{1}{2}\ln(x^2+4) + C_1\right) e^x + \left(\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C_2\right) x e^x$$

FINAL ANSWER

$$\begin{cases} y_1(x) = e^x \\ y_2(x) = xe^x \end{cases}$$

$y_1'(x) = e^x$

$y_2'(x) = xe^x + e^x$

(*)
$$\begin{cases} C_1'(x)\, y_1(x) + C_2'(x)\, y_2(x) = 0 \\ c_1'(x)\, y_1'(x) + C_2'(x)\, y_2'(x) = f(x) \end{cases}$$

$$\begin{cases} C_1'(x) e^x + C_2'(x)\, x e^x = 0 & \Big| : e^x \\ C_1'(x) e^x + C_2'(x)(x+1) e^x = \frac{e^x}{x^2+4} & \Big| e^x \end{cases}$$

$$\begin{cases} C_1' + C_2' \cdot x = 0 \\ C_1' + C_2' \cdot (x+1) = \frac{1}{x^2+4} \end{cases} \Bigg|^{-}$$

$C_2' = \frac{1}{x^2+4} \Rightarrow C_2(x) = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C_2$

$C_1' = -x C_2' = -\frac{x}{x^2+4} \Rightarrow C_1(x) = -\frac{1}{2}\ln(x^2+4) + C_1$

$$y'' + py' + qy = f(x) \qquad \text{НЕ} \quad y'' + py' + qy = 0$$

$$y = y_1(x)$$

$$y = y_2(x)$$

$$y = C_1(x)\, y_1(x) + C_2(x)\, y_2(x)$$

$$y' = C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2 y_2' = C_1 y_1' + C_2 y_2' + \underbrace{C_1' y_1 + C_2' y_2}_{=0} \quad \mathrm{I}$$

$$\begin{cases} y' = C_1 y_1' + C_2 y_2' \\ y'' = C_1'' y_1 + 2 C_1' y_1' + C_1 y_1'' + C_2'' y_2 + 2 C_2' y_2' + C_2 y_2'' \end{cases}$$

$$\Rightarrow y'' = C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2'$$

$$f(x) = y'' + py' + qy = C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2' + p C_1 y_1' + p C_2 y_2' + q C_1 y_1 + q C_2 y_2$$

$$= C_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{=0} + C_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{=0} + C_1' y_1' + C_2' y_2' =$$

$$= C_1' y_1' + C_2' y_2' \quad \mathrm{II}$$

$$\begin{aligned} \mathrm{I} \quad & \begin{cases} C_1' y_1 + C_1' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x) \end{cases} \\ \mathrm{II} \quad & \end{aligned}$$

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x) \end{cases}$$

$$C_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{f y_2}{W}$$

CRAMER FORMULAS

$$C_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{f y_1}{W}$$

$$W = \underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\text{WROŃSKI DETERMINANT (WRONSKIAN)}} = y_1 y_2' - y_1' y_2$$

$$y'' - 3y' + 2y = \underline{1}e^{-x}$$

PREDICTION:
$y = Ae^{-x}$
$y' = -Ae^{-x}$
$y'' = Ae^{-x}$
$y'' - 3y' + 2y = \underline{\underline{6A}}e^{-x}$
$6A = 1$

(HG) $y'' - \underbrace{3}y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$
$$\parallel$$
$$(r-1)(r-2)$$

$$y = C_1 \underbrace{e^x}_{y_1(x)} + C_2 \underbrace{e^{2x}}_{y_2(x)}$$

$$C_1' = -\frac{f y_2}{W} = -\frac{e^{-x} e^{2x}}{e^{3x}} = -e^{-2x}$$

$$C_2' = \frac{f y_1}{W} = \frac{e^{-x} e^{x}}{e^{3x}} = e^{-3x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

$$C_1(x) = \int -e^{-2x}\,dx = \frac{1}{2}e^{-2x} + C_1$$

$$C_2(x) = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C_2$$

FINAL ANSWER #1

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x) = \left(\frac{1}{2}e^{-2x} + C_1\right)e^x + \left(-\frac{1}{3}e^{-3x} + C_2\right)e^{2x}$$

$$= \frac{1}{2}e^{-x} + C_1 e^x + \left(-\frac{1}{3}\right)e^{-x} + C_2 e^{2x} =$$

$$= C_1 e^x + C_2 e^{2x} + \frac{1}{6}e^{-x}$$ ← FINAL ANSWER #2

EX 1. Q:

$$\boxed{\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) \end{cases}}$$

DERIVATIVE

$$\Rightarrow \quad x''(t) = y'(t) = -x(t)$$

$$\Rightarrow y''(t) = -x'(t) = -y(t)$$

$$\boxed{x'' + x = 0} \qquad r^2 + 1 = 0$$

$$r = \pm i$$

$$y'' + y = 0$$

$$y(t) = D_1 \cos t + D_2 \sin t$$

ANSWER:

$$\begin{cases} x(t) = C_1 \cos t + C_2 \sin t \\ y(t) = x'(t) = -C_1 \sin t + C_2 \cos t \end{cases}$$

EX 2. Q:

$$\boxed{\begin{cases} x' = 2x + 3y \\ y' = x - 2y \end{cases}} \Rightarrow x'' = 2x' + 3y'$$

$$\Rightarrow y = \frac{x' - 2x}{3}$$

$$x'' = 2x' + 3(x - 2y) =$$

$$= 2x' + 3x - 6y$$

$\times 2$

$$2x' = 4x + 6y$$

$$x'' + 2x' = 2x' + 3x \cancel{-6y} + 4x + \cancel{6y} = 2x' + 7x \Rightarrow \boxed{x'' = 7x}$$

$$x'' - 7x = 0$$

$$x = C_1 e^{\sqrt{7}t} + C_2 e^{-\sqrt{7}t}$$

$$y = C_1 \frac{\sqrt{7} - 2}{3} e^{\sqrt{7}t} + C_2 \frac{-\sqrt{7} - 2}{3} e^{-\sqrt{7}t}$$

EX.

$$\boxed{x^{(3)}(t) - 3x'(t) + 2x(t) = e^t}$$

$$x^{(3)}(t) = 3x'(t) - 2x(t) + e^t$$

DEFINE:

$y(t) = x'(t)$

$z(t) = y'(t) = x''(t)$

$z'(t) = x^{(3)}(t)$

$$\begin{cases} x'(t) = y(t) \\ y'(t) = z(t) \\ z'(t) = 3y(t) - 2x(t) + e^t \end{cases}$$

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$

$$\bar{X}'(t) = A \cdot \bar{X}(t) + B$$

HE $\quad x^{(3)} - 3x' + 2x = 0$

$r^3 - 3r + 2 = 0$

$(r-1)(r^2 + r - 2) =$

$(r-1)(r-1)(r+2)$

$\vdots$

$$\bar{X}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$$

$$x'(t) = a\, x(t) \iff x(t) = C e^{at} = C\left(1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \dots + \frac{a^n t^n}{n!} + \dots\right)$$

$$\bar{X}'(t) = \underbrace{A}_{(n \times n)\text{ MATRIX}} \bar{X}(t) \iff X(t) = C e^{tA} = C\left(I + t A + \frac{t^2}{2!} A^2 + \dots + \frac{t^n}{n!} A^n + \dots\right)$$

$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

DEFINITION (OF e^{matrix})

EX COMPUTE e^A FOR

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$A = \begin{bmatrix}1 & 3\\ 7 & 9\end{bmatrix} \Rightarrow e^A = ?$ (HARD TO COMPUTE)

$A \cdot B$ MATRIX MULTIPLICATION

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix} & 2\\ & \end{bmatrix} \quad 2 = 1\cdot 1 + 1 \cdot 1$$

$$\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix} & 3\\ & \end{bmatrix} \quad 3 = 1\cdot 1 + 2 \cdot 1$$

$$A^2 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

$$\vdots$$

$$A^n = \begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$$

ANSWER: $e^A = \begin{bmatrix}e & e\\ 0 & e\end{bmatrix}$

$$e = 1 + 1 + \frac{1}{2} + \dots + \frac{1}{(n-1)!} + \dots$$

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots + \frac{1}{n!}A^n + \dots =$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} + \dots + \frac{1}{n!}\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix} + \dots = \begin{bmatrix}\underbrace{1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots}_{e} & \underbrace{1 + \frac{2}{2!} + \dots + \frac{n}{n!}}_{e} \\ 0 & \underbrace{1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} \dots}_{e}\end{bmatrix}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \Rightarrow e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

DIFFERENCE*) EQUATIONS

*) (AND SIMILAR DISCRETE)

$12 = 2\cdot 6 - 2 + 2$ $\quad 20 = 2\cdot 12 - 6 + 2$ $\quad 30 = 2\cdot 20 - 12 + 2$ $\quad 42 = 2\cdot 30 - 20 + 2$

$1\cdot 2 \quad 2\cdot 3 \quad 3\cdot 4 \quad 4\cdot 5 \quad 5\cdot 6 \to 6\cdot 7$

EX. $(a_n) = (\boxed{2, 6}, 12, 20, 30 \to 42 \quad 56 \quad 72$

$(b_n) = \quad 6-2=4 \quad 6 \quad 8 \quad 10 \to 12$

$(c_n) = \quad 2 \quad 2 \quad 2 \sim 2$

$0 \quad 0 \to 0$

DIFFERENCE OPERATOR (d_n)

$a_1 = 2$
$a_2 = 6$
$a_3 = 12$
$\vdots$
$a_n = n(n+1) = n^2 + n$

$b = \Delta a$ I.E. $b_n = a_{n+1} - a_n$

$c = \Delta b = \Delta^2 a \qquad c_n = b_{n+1} - b_n =$

$= (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) =$

$= a_{n+2} - 2a_{n+1} + a_n$

$d = \Delta c = \Delta^3 a$

$$\begin{cases} a_1 = 2 \\ a_2 = 6 \\ a_{n+2} - 2a_{n+1} + a_n = 2 \end{cases} \quad \text{for } n \geq 1$$

$a_{n+2} = 2a_{n+1} - a_n + 2$

$$\begin{cases} a_1 = 1 & 1 \\ a_2 = 1 & 3 \\ \boxed{a_{n+2} = a_{n+1} + a_n} \end{cases}$$

$(\boxed{1, 1}, 2, 3, 5, 8, 13, 21, 34, \dots)$ ②

$(\boxed{1, 3}, 4, 7, 11, 18, 29, 47, \dots)$

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

← NEXT WEEK

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

RECURSIVE DEFINITION

MAY 25TH

← INITIAL CONDITIONS

$\boxed{a_n = n^2 + n}$

EXPLICIT DEFINITION

$1\cdot 2 \quad 2\cdot 3 \quad 3\cdot 4 \quad 4\cdot 5 \quad 5\cdot 6$

$(2, 6, 12, 20, 30, \dots)$

$a_{1000} = 1{,}001{,}000$

$$\begin{cases} a_1 = 2 \\ a_{n+1} = a_n + 2n + 2 \end{cases}$$

ORDER 1 (for $n \geq 1$)

$a_2 = a_1 + 4 \quad a_4 = a_3 + 8$

$(2, 6, 12, 20, 30, \dots)$

$a_3 = a_2 + 6$

$a_5 = a_4 + 10$

$$\begin{cases} a_1 = 2 \\ a_2 = 6 \\ a_{n+2} = 2a_{n+1} - a_n + 2 \end{cases}$$

(for $n \geq 1$)

ORDER 2

$2\cdot 6 - 2 + 2 \qquad 2\cdot 20 - 12 + 2$

$(2, 6, 12, 20, 30, \dots)$

$2\cdot 12 - 6 + 2$

$$\begin{cases} a_1 = 2 \\ a_2 = 6 \\ a_3 = 12 \\ a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n \end{cases}$$

(for $n \geq 1$)

ORDER 3

$3\cdot 12 - 3\cdot 6 + 2$

$(2, 6, 12, 20, 30, \dots)$

$3\cdot 20 - 3\cdot 12 + 6$

$$(*)\begin{cases} a_0 = 1 \\ a_1 = 2 \quad 3 \quad 4 \\ \underline{a_n = 6a_{n-1} - 8a_{n-2}} \quad (\text{for } n \geq 2) \end{cases}$$

$a_2 = 6a_1 - 8a_0 = 6 \cdot 2 - 8 \cdot 1 = 4$ $\quad 6 \cdot 3 - 8 \cdot 1 = 10$ $\quad 6 \cdot 4 - 8 \cdot 1 = 16$

$a_3 = 6a_2 - 8a_1 = 6 \cdot 4 - 8 \cdot 2 = 8$ $\quad 6 \cdot 10 - 8 \cdot 3 = 36$ $\quad 6 \cdot 16 - 8 \cdot 4 = 64$

$a_4 = 6a_3 - 8a_2 = 6 \cdot 8 - 8 \cdot 4 = 16$ $\quad 6 \cdot 36 - 8 \cdot 10 = 136$ $\quad 6 \cdot 64 - 8 \cdot 16 = 256$

$\vdots$

$(\boxed{1, 2}, 4, 8, 16, \ldots)$ $\quad (\boxed{1, 3}, 10, 36, 136, \ldots)$ $\quad (\boxed{1, 4}, 16, 64, 256, \ldots)$

$a_n = \underline{2^n}$ $\quad a_n \neq 3^n \quad a_n = \frac{1}{2} 2^n + \frac{1}{2} 4^n$ $\quad a_n = \underline{4^n}$

IF $a_n = a_0 q^n$ SATISFIES $(*)$, $a_0, q \neq 0$

FACT: EVERY SEQUENCE $\underline{a_n = A \cdot 2^n + B \cdot 4^n}$ SATISFIES $(*)$

$$(*) \quad a_0 q^n = 6a_0 q^{n-1} - 8a_0 q^{n-2} \quad \Big| : (a_0 q^{n-2})$$

$$q^2 = 6q - 8 \qquad q^2 - 6q + 8 = 0 \implies q = 2 \text{ OR } q = 4$$

$$(q-2)(q-4)$$

LET $\underbrace{a_n = A \cdot 2^n + B \cdot 4^n}$

$a_{n-1} = A \cdot 2^{n-1} + B \cdot 4^{n-1} = \frac{1}{2} A 2^n + \frac{1}{4} B \cdot 4^n$

$a_{n-2} = A \cdot 2^{n-2} + B\, 4^{n-2} = \frac{1}{4} A 2^n + \frac{1}{16} B \cdot 4^n$

$RHS = 6 a_{n-1} - 8 a_{n-2} = 6\left(\frac{1}{2} A 2^n + \frac{1}{4} B \cdot 4^n\right) - 8\left(\frac{1}{4} A \cdot 2^n + \frac{1}{16} B 4^n\right) =$

$= 3 A \cdot 2^n + \frac{3}{2} B\, 4^n - 2 A \cdot 2^n - \frac{1}{2} B 4^n =$

$= A \cdot 2^n + B \cdot 4^n = LHS$

$\begin{cases} a_0 = A \cdot 2^0 + B\, 4^0 = A + B \\ a_1 = A \cdot 2^1 + B \cdot 4^1 = 2A + 4B \end{cases} \Rightarrow B = a_0 - A$

$a_1 = 2A + 4a_0 - 4A \Rightarrow \begin{cases} A = \dfrac{4a_0 - a_1}{2} \\ B = \dfrac{a_1 - 2a_0}{2} \end{cases}$

$B = a_0 - A$

$\begin{cases} a_0 = 1 \\ a_1 = 2 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 0 \end{cases}$ $a_n = 2^n$

$\begin{cases} a_0 = 1 \\ a_1 = 3 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \end{cases}$ $a_n = \dfrac{2^n + 4^n}{2}$

$\begin{cases} a_0 = 1 \\ a_1 = 4 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 1 \end{cases}$ $a_n = 4^n$

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{cases} \iff a_{n+2} = a_{n+1} + a_n \quad (*) \quad \text{for } n \geq 1$$

$a_n = a_{n+2} - a_{n+1}$

$1/1 = 1$
$2/1 = 2$
$3/2 = 1.5$
$5/3 = 1.666\ldots$
$8/5 = 1.6$
$13/8 = 1.625$
$21/13 = 1.616\ldots$
$34/13 = \ldots$

$5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots$

$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$

$\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2}$

$q^2 = q + 1$ (CHARACTERISTIC EQUATION)

$q^2 - q - 1 = 0$

$a = 1; b = c = -1, \Delta = b^2 - 4ac = (-1)^2 - 4 \cdot 1(-1) = 1 + 4 = 5$

$q_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{1+\sqrt{5}}{2}$; $q_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{1-\sqrt{5}}{2}$

GEN. SOL. OF $(*)$:

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \end{cases} \Rightarrow \ldots \Rightarrow \begin{cases} A = \frac{1}{\sqrt{5}} \\ B = -\frac{1}{\sqrt{5}} \end{cases}$$

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0$

$\boxed{\frac{1+\sqrt{5}}{2}} \approx 1.618\ldots > 1$

$\frac{1-\sqrt{5}}{2} \approx -0.618\ldots \in (-1; 1)$

$$\begin{cases} a_0 = 1 \\ a_1 = 3 \\ a_{n+2} = 4(a_{n+1} - a_n) \end{cases}$$

$4 \cdot \Delta a_n$

a_n (1, 3, 8, 20, 48, 112, 256 ...)

$2 \cdot \frac{1}{2}$ 3 1 4 2 5 4 6 8 7 16 8 32

$\Delta a_n = (2, 5, 12, 28, 64, \ldots)$

$8 \cdot 32 = 256$

$y''(x) - 4y'(x) + 4y(x) = 0 \leadsto a_{n+2} - 4a_{n+1} + 4a_n = 0$

$$q^2 - 4q + 4 = 0$$

$\Delta = 16 - 16 = 0$

$q_1 = q_2 = 2$

$\cancel{a_n = A \cdot 2^n + B \cdot 2^n}$

$$q^2 - 4q + 4 = (q-2)^2$$

$$a_n = A \cdot 2^n + B\, n\, 2^n$$

$$a_n = 2^n + \frac{1}{2} n 2^n = (n+2) 2^{n-1}$$

$y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$r_1 = r_2 = 2$

$y(x) = C_1 e^{2x} + C_2 x e^{2x}$

$(e^x)' = e^x$

$$\begin{cases} a_0 = A \cdot 2^0 + B \cdot 0 \cdot 2^0 = A \\ a_1 = A \cdot 2^1 + B \cdot 1 \cdot 2^1 = 2A + 2B \end{cases}$$

$$\begin{cases} A = a_0 \\ B = \frac{a_1 - 2a_0}{2} \end{cases}$$

$$\begin{cases} a_0 = 1 \\ a_1 = 3 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = \frac{1}{2} \end{cases}$$

$\Delta(2^n) = 2^n$

1 2 4 8 16 32 ....

Δ: 1 2 4 8 16

$a_{n+1} - a_n = a_n$

$\Rightarrow a_{n+1} = 2 \cdot a_n$

GENERATING FUNCTIONS

$(1,1,1,1,\dots,1,\dots) \mapsto 1+1x+1\cdot x^2+1\cdot x^3+\dots = \frac{1}{1-x}$ FOR $|x|<1$

$(1,2,4,8,16,\dots) \mapsto 1+2\cdot x+4x^2+8x^3+\dots =$
$= 1+2x+(2x)^2+(2x)^3+\dots = \frac{1}{1-2x}$ FOR $|x|<\frac{1}{2}$

$x\in(-\frac{1}{2};\frac{1}{2})$

$(a_n) \mapsto \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{A(x)} = a_0+a_1x+a_2x^2+\dots$

$A(x)$ ← GENERATING FUNCTION OF THE SEQUENCE (a_n)

$0.2+0.02+0.002+\dots$

$x = 0.22222\dots$

$(-)\quad 10x = 2.22222\dots = 2+x$

$9x = 10x - x = 2$

$\Downarrow$

$x = \frac{2}{9}$

Ex.1

$a_n = a_0 q^n \Rightarrow A(x) = \sum_{n=0}^{\infty} a_0 q^n x^n =$
$= a_0 \sum_{n=0}^{\infty} (qx)^n = \frac{a_0}{1-qx}$

$S = a_0 + a_0q + a_0q^2 + \dots + a_0q^n + \dots$

$qS = a_0q + a_0q^2 + a_0q^3 + \dots$

$S = a_0 + qS \Rightarrow (1-q)S = S - qS = a_0$

$\Downarrow$

$S = \boxed{\frac{a_0}{1-q}}$

IF $|q|<1$ ← FOR WHAT q?

$\infty - 2\infty = ?$

$S = 1+2+4+8+\dots$

$2S = 2+4+8+\dots$

$\Rightarrow S - 2S = 1$

$\Downarrow$

$S = -1$

Ex.2 $a_n = n$

$A(x) = x+2x^2+3x^3+4x^4+\dots =$
$= (x+x^2+x^3+x^4+\dots)+$
$(x^2+x^3+x^4+\dots)+$
$(x^3+x^4+\dots)+$
$(x^4+\dots)+\dots = \frac{x}{1-x}+\frac{x^2}{1-x}+\frac{x^3}{1-x}+\dots = \frac{x}{(1-x)^2}$

$a_0 = \frac{x}{1-x}$

$q = x$

Ex 2.

cont. $A(x) = \underbrace{x + 2x^2 + 3x^3 + 4x^4 + \ldots}_{\sum_{n=0}^{\infty} n x^n} = x(1 + 2x + 3x^2 + 4x^3 + \ldots) =$ WHY CAN WE DO THAT?

$= x\left(x' + (x^2)' + (x^3)' + (x^4)' + \ldots\right) =$

$= x\left(x + x^2 + x^3 + x^4 + \ldots\right)' = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$

$|x| < 1$

$$\begin{cases} a_0 = 1 \\ a_1 = 3 \\ a_{n+2} = 4(a_{n+1} - a_n) \end{cases}$$

$8 = 4(3-1)$

$20 = 4(8-3)$

$(a_n) = (1, 3, 8, 20, 48, \ldots)$

$(x^2 + x^3)' \quad 2x + 3x^2$

$= (x^2)' + (x^3)'$

$A(x) = 1 + 3x + 8x^2 + 20x^3 + 48x^4 + \ldots =$

$= 1 + 3x + 4(3-1)x^2 + 4(8-3)x^3 + 4(20-8)x^4 + \ldots =$

$= 1 + 3x + 4 \cdot 3x^2 - 4x^2 + 4 \cdot 8x^3 - 4 \cdot 3x^3 + 4 \cdot 20x^4 - 4 \cdot 8x^4 + \ldots =$

$= 1 + 3x + 4(3x^2 + 8x^3 + 20x^4 + \ldots) - 4(x^2 + 3x^3 + 8x^4 + \ldots) =$

$= 1 + 3x + 4x(\cancel{1} + 3x + 8x^2 + 20x^3 + \ldots) - 4x^2(1 + 3x + 8x^2 + \ldots) =$

$= 1 + 3x + 4x(A(x) - 1) - 4x^2 A(x) = 1 - x + (4x - 4x^2)A(x)$

LHS

$(1 - 4x + 4x^2)A(x) = 1 - x \implies A(x) = \frac{1-x}{1-4x+4x^2} = \frac{(1-2x) + x}{(1-2x)^2} = \frac{1}{1-2x} + \frac{1}{2} \cdot \frac{2x}{(1-2x)^2} =$

$\sum_{n=0}^{\infty} \boxed{a_n} x^n = \sum_{n=0}^{\infty} (2x)^n + \frac{1}{2}\sum_{n=0}^{\infty} n(2x)^n = \sum_{n=0}^{\infty} \boxed{\left(2^n + \frac{1}{2} n 2^n\right)} x^n$

$$\begin{cases} a_0 = 1 \\ a_1 = 3 \\ a_{n+1} = 4a_{n+1} - 4a_n \end{cases} \quad \Big| \cdot x^{n+2}$$

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = A(x)$$

$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = a_1 x + a_2 x^2 + a_3 x^3 + \dots = A(x) - a_0$$

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = a_2 x^2 + a_3 x^3 + \dots = A(x) - a_0 - a_1 x$$

$$a_{n+2} x^{n+2} = 4a_{n+1} x^{n+2} - 4a_n x^{n+2} = 4x \cdot a_{n+1} x^{n+1} - 4x^2 a_n x^n$$

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = 4x \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - 4x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$A(x) - 1 - 3x = 4x\,(A(x) - 1) - 4x^2 A(x)$$

$A(x) \to$ LHS

no $A(x) \to$ RHS

$$A(x)\left[1 - 4x + 4x^2\right] = 1 + 3x - 4x$$

$$A(x) = \frac{1-x}{1-4x+4x^2} = \frac{(1-2x)+x}{(1-2x)^2} = \frac{1}{1-2x} + \frac{1}{2}\,\frac{2x}{1-2x} = \dots$$

$$A(x) \,\|\, \sum a_n x^n$$

$$= \sum_{n=0}^{\infty} \left(2^n + \frac{1}{2} n 2^n\right) x^n$$

$$a_n = 2^n + \frac{1}{2} n 2^n = (n+2) \cdot 2^{n-1}$$