$$\begin{cases} x''(t) - 3x'(t) + 2x(t) = 0 \\ x(0) = 2 \\ x'(0) = 3 \end{cases}$$

$$r^2 - 3r + 2 = 0$$

$$(r-1)(r-2)$$

$$\Delta = b^2 - 4ac = 9 - 8 = 1$$

$$\sqrt{\Delta} = 1$$

$$r_1 = 1 \qquad r_2 = 2$$

$$\boxed{x(t) = C_1 e^{t} + C_2 e^{2t}}$$

$$x(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2$$

$$x'(t) = C_1 e^t + C_2 2e^{2t}$$

$$x'(0) = C_1 + 2C_2$$

$$\begin{cases} C_1 + C_2 = x(0) = 2 \\ C_1 + 2C_2 = x'(0) = 3 \end{cases}$$

$$\begin{cases} C_2 = 1 \\ C_1 = 2 - C_2 = 1 \end{cases}$$

$$x(t) = e^t + e^{2t}$$

LAPLACE TRANSFORMATION

$$f(t) \overset{L}{\rightsquigarrow} \bar{f}(s)$$

$$\underbrace{L[f] = \bar{F}(s)}_{\text{NOTATION}} \underset{\text{DEFINITION}}{=} \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

Ex) $\begin{cases} x'(t) = 2x(t) \\ x(0) = 3 \end{cases} \quad \Big|\, e^{-st}$

$x'(t)\, e^{-st} = 2\, x(t)\, e^{-st} \quad \Big|\, \int \ldots$

$$\int_0^{+\infty} x'(t) e^{-st} dt = 2 \underbrace{\int_0^{+\infty} x(t) e^{-st}\, dt}_{= \bar{x}(s)}$$

$$\| $$

$\begin{vmatrix} x'(t) = x'(t) & y(t) = e^{-st} \\ x(t) = x(t) & y'(t) = -s\, e^{-st} \end{vmatrix}$

$$\|$$

$$\underbrace{x(t) e^{-st} \Big|_{t=0}^{+\infty}}_{} - \int_0^{+\infty} x(t)(-s) e^{-st}\, dt = \bar{x}(s)$$

$$= \boxed{\lim_{M \to +\infty} (x(M) e^{-Ms})} - \underbrace{x(0)}_{=3} + s \int_0^{+\infty} x(t) e^{-st} dt = -3 + s\,\bar{x}(s)$$

$x'(t) = 2x(t) \quad | \int$

$\Downarrow$

$x(t) - x(0) = 2 \int_0^t x(w)\, dw$

$\| $

$\int_0^t x'(w)\, dw$

$\boxed{a \mapsto a - 1}$

$3x + 1 = 7$

$\Leftarrow$ TR. $\Rightarrow \Downarrow$

$(3x + 1) - 1 = 7 - 1$

$3x = 6$

$\boxed{a \mapsto \frac{a}{3}}$

TR. $\Downarrow$

$\frac{3x}{3} = \frac{6}{3}$

$x = 2$

$\int x' y = xy - \int x y'$

$(*) \quad -3 + s\,\bar{x}(s) = 2\,\bar{x}(s)$

$\Downarrow$

$(s-2)\,\bar{x}(s) = 3 \Rightarrow \bar{x}(s) = \frac{3}{s-2}$

$x(t) = 3e^{2t}$

$$\bar{x}(s) = \int_0^{+\infty} x(t)\, e^{-st}\, dt$$

$$L[a\,x(t)](s) = a \cdot L[x(t)](s)$$

$$L[x+y] = L[x] + L[y]$$

EX1. LET $x(t) = 1$

$$\bar{x}(s) = \boxed{\int_0^{+\infty} 1 \cdot e^{-st}\, dt} = \lim_{M\to+\infty} \int_0^M e^{-st}\, dt = \lim_{M\to\infty} \left[\frac{e^{-st}}{-s} \right]_{t=0}^{t=M} =$$

$$= \lim_{M\to\infty} \left[\frac{e^{-Ms}}{-s} - \frac{e^0}{-s} \right] = \frac{1}{s} - \frac{1}{s} \lim_{M\to\infty} \left(e^{-Ms}\right) =$$

$$\lim_{M\to\infty} \left(e^{-Ms}\right) = 0 \text{ for } s>0$$

$$= \begin{cases} \boxed{1/s} & \text{if } s>0 \\ \boxed{+\infty \quad \text{if } s \le 0} \end{cases}$$

→ NOT VERY IMPORTANT

SIDE REMARK

$7 \cdot 8 = 56$

$(a+b)^2 = a^2 + 2ab + b^2$

$(x^3)' = 3x^2 \qquad (x^n)' = n\,x^{n-1}$

$\int x^3\, dx = \frac{x^4}{4} + C$

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$

$n \neq -1$

EX2. LET $x(t) = e^{at}$

$$\bar{x}(s) = \int_0^{+\infty} e^{at}\, e^{-st}\, dt = \int_0^{+\infty} e^{\overbrace{-(s-a)t}^{at-st}}\, dt = \frac{1}{s-a}$$

EX. same s-a

(for $s>a$)

$$\bar{x}(s) = \frac{3}{s-2} \Rightarrow x(t) = 3e^{2t}$$

WHY

$f(t)$	$\bar{f}(s)$
1	$\boxed{1/s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

Ex 3. $f(t) = \cos(bt) = \frac{1}{2} e^{ibt} + \frac{1}{2} e^{-ibt}$

EULER
$$\begin{cases} e^{iy} = \cos y + i \sin y \\ e^{-iy} = \cos y - i \sin y \end{cases}$$
$+$
$$e^{iy} + e^{-iy} = 2\cos y$$

$$L[\cos(bt)] = \frac{1}{2} L[e^{ibt}] + \frac{1}{2} L[e^{-ibt}] =$$

Ex. 2
$$= \frac{1}{2} \frac{1}{s - ib} + \frac{1}{2} \frac{1}{s + ib} =$$

$$= \frac{1}{2} \frac{(s+ib) + (s-ib)}{(s-ib)(s+ib)} = \frac{s}{s^2 + b^2}$$

Ex 4. $f(t) = t \Rightarrow \bar{f}(s) = \int_0^{+\infty} t e^{-st} dt = \dots = \frac{1}{s^2}$

$$\boxed{L[f'(t)] = s \cdot L[f(t)] - f(0)}$$

$$L[f''(t)] = s \cdot L[f'(t)] - f'(0) =$$
$$= s(s L[f(t)] - f(0)) - f'(0) =$$
$$= s^2 L[f(t)] - s f(0) - f'(0)$$

$$\begin{cases} \boxed{x'' - 3x' + 2x = 0} \xrightarrow{\mathcal{L}} \mathcal{L}[x'' - 3x' + 2x] = \mathcal{L}[0] = 0 \\ x(0) = 2 \\ x'(0) = 3 \end{cases}$$

$$\mathcal{L}[x''] - 3\mathcal{L}[x'] + 2\mathcal{L}[x] \quad \mathcal{L}[x] = \bar{x}(s)$$

$$\left(s^2 \bar{x}(s) - \underset{x(0)}{2}s - \underset{x'(0)}{3}\right) - 3\left(s\bar{x}(s) - \underset{x(0)}{2}\right) + 2\bar{x}(s)$$

$$(s^2 - 3s + 2)\,\bar{x}(s) - 2s - 3 + 6$$

$$(s^2 - 3s + 2)\,\bar{x}(s) = 2s - 3$$

$$\bar{x}(s) = \frac{2s-3}{s^2-3s+2} = \frac{1}{s-1} + \frac{1}{s-2} = \mathcal{L}[e^t] + \mathcal{L}[e^{2t}]$$

$$\Downarrow$$

$$x(t) = e^t + e^{2t}$$

$$\frac{2s-3}{s^2-3s+2} = \frac{2s-3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} = \frac{A(s-2) + B(s-1)}{(s-1)(s-2)} \quad \Big| \cdot (s-1)(s-2)$$

$$2s - 3 = A(s-2) + B(s-1)$$

$s=1$: $2 - 3 = A(1-2) + B(1-1) = -A \Rightarrow A = 1$

$s=2$: $4 - 3 = A \cdot 0 + B \cdot 1 \Rightarrow B = 1$

$$\begin{cases} x'' - 4x' + 3x = 3 \\ x(0) = -2 \\ x'(0) = 3 \end{cases}$$

$t^2 - 4t + 3 = 0$

$L[3] = \frac{3}{s}$

$L[x'(t)] = s\,L[x(t)] - x(0) = s\bar{x}(s) + 2$, where $L[x(t)] = \bar{x}(s)$

$L[x''(t)] = s^2 L[x(t)] - s\,x(0) - x'(0) = s^2\bar{x}(s) + 2s - 3$

$s^2 - 4s + 3 = 0$
$\Delta = 16 - 12 = 4$

$L[x''] - 4L[x'] + 3L[x] = L[3]$

$(s^2\bar{x}(s) + 2s - 3) - 4(s\bar{x}(s) + 2) + 3\bar{x}(s) = \frac{3}{s}$

$s^2\bar{x}(s) - 4s\bar{x}(s) + 3\bar{x}(s) = \frac{3}{s} - 2s + 3 + 8 = \frac{3}{s} - 2s + 11 = \frac{-2s^2 + 11s + 3}{s}$

$[s^2 - 4s + 3]\bar{x}(s) \Rightarrow \bar{x}(s) = \frac{-2s^2 + 11s + 3}{s(s^2 - 4s + 3)} = \frac{-2s^2 + 11s + 3}{s(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3}$

$* \; s(s-1)(s-3)$

$-2s^2 + 11s + 3 = A(s-1)(s-3) + Bs(s-3) + Cs(s-1)$

$s=0 \quad 3 = A(-1)(-3) \Rightarrow A = \frac{3}{3} = 1$

$s=1 \quad -2 + 11 + 3 = B \cdot 1 \cdot (-2) \Rightarrow B = \frac{12}{-2} = -6$

$s=3 \quad -18 + 33 + 3 = C \cdot 3 \cdot 2 \Rightarrow C = \frac{18}{6} = 3$

$\frac{1}{s} - \frac{6}{s-1} + \frac{3}{s-3}$

$L[be^{at}] = \frac{b}{s-a}$

$L[1 - 6e^t + 3e^{3t}]$

ANS: $x(t) = 1 - 6e^t + 3e^{3t}$

$$\begin{cases} x'' - 4x' + 4x = -4 \\ x(0) = -2 \\ x'(0) = 3 \end{cases} \quad \mathcal{L}$$

$$as^2 + bs + c = 0$$
$$s^2 - 4s + 4 = 0$$
$$b^2 - 4ac = \Delta = 4^2 - 4 \cdot 4 \cdot 1 = 0$$

$$(s^2 \bar{x}(s) + 2s - 3) - 4(s\bar{x}(s) + 2) + 4\bar{x}(s) = -\frac{4}{s}$$

$$(s^2 - 4s + 4)\bar{x}(s) = -\frac{4}{s} - 2s + 3 + 8 = \frac{-2s^2 + 11s - 4}{s}$$

$$\bar{x}(s) = \frac{-2s^2 + 11s - 4}{s(s^2 - 4s + 4)} = \frac{-2s^2 + 11s - 4}{s(s-2)^2} = \frac{A}{s} + \frac{B}{(s-2)^1} + \frac{C}{(s-2)^2} \quad \Big| \cdot s(s-2)^2$$

$$-2s^2 + 11s - 4 = A(s-2)^2 + Bs(s-2) + Cs$$

$s=0$ $\quad -4 = A(-2)^2 \Rightarrow A = \frac{-4}{4} = -1$

$s=2$ $\quad -8 + 22 - 4 = C \cdot 2 \Rightarrow C = \frac{10}{2} = 5$

$s=1$ $\quad -2 + 11 - 4 = A(-1)^2 + B\,1(1-2) + C \cdot 1$

$$B = \frac{5 - A - C}{-1} = \frac{1}{-1} = -1$$

$$= -\frac{1}{s} + \left(-\frac{1}{s-2}\right) + \frac{5}{(s-2)^2}$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$$

$n = 1 ; a =$

$$= \mathcal{L}[-1 - e^{2t} + 5te^{2t}]$$

ANS: $x(t) = 5te^{2t} - e^{2t} - 1$