On recursive sequences

1. Homogeneous linear equations

We consider the following equation:

$$a_{n+k} + c_{k-1}a_{n+k-1} + \ldots + c_1 a_{n+1} + c_0 a_n = 0$$

where c_i are constant numbers.

That equation is satisfied by the geometric sequences $a_n = r^n$ if r is a root of the **characteristic polynomial**

$$w(r) = r^k + c_{k-1}r^{k-1} + \ldots + c_1 r + c_0,$$

and all their linear combinations.

Ex. 1. Find all sequences satisfying $a_{n+2} = 5a_{n+1} - 6a_n$.

Solution. We have the equation $a_{n+2} - 5a_{n+1} + 6a_n = 0$ and the corresponding characteristic polynomial $r^2 - 5r + 6$ with roots 2 amd 3. We have special solutions $a_n = 2^n$ i $a_n = 3^n$ and the general one $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$.

Let's **check** the answer. If $a_n = C_1 2^n + C_2 3^n$ then

$$a_{n+1} = C_1 2^{n+1} + C_2 3^{n+1} = 2C_1 2^n + 3C_2 3^n$$

$$a_{n+2} = C_1 2^{n+2} + C_2 3^{n+2} = 4C_1 2^n + 9C_2 3^n$$

$$a_{n+2} - 5a_{n+1} + 6a_n = 4C_1 2^n + 9C_2 3^n - 10C_1 2^n - 15C_2 3^n + 6C_1 2^n + 6C_2 3^n = 0,$$

because $4 - 10 + 6 = 9 - 15 + 6 = 0$. Verified.

Ex. 2. Find all sequences satisfying $a_{n+2} = 2(a_{n+1} - a_n)$.

Solution. The equation is $a_{n+2} - 2a_{n+1} + 2a_n = 0$. The characteristic polynomial is $r^2 - 2r + 2$ with roots $1 + i$ and $1 - i$. We thus have special solutions $a_n = (1 + i)^n$ and $a_n = (1 - i)^n$ and the general one $a_n = C_1(1 + i)^n + C_2(1 - i)^n$.

Remark. If you prefer a fully real answer you may use the trigonometric form of complex numbers and de Moivre's formula:

$$1 \pm i = \sqrt{2}\left(\cos\frac{\pi}{4} \pm \sin\frac{\pi}{4} \cdot i\right)$$

$$(1 \pm i)^n = (\sqrt{2})^n \left(\cos\frac{n\pi}{4} \pm \sin\frac{n\pi}{4} \cdot i\right)$$

$$a_n = C_1(1 + i)^n + C_2(1 - i)^n = (\sqrt{2})^n \left((C_1 + C_2)\cos\frac{n\pi}{4} + i(C_1 - C_2)\sin\frac{n\pi}{4}\right),$$

which can be written as

$$a_n = D_1 \cdot (\sqrt{2})^n \cos\frac{n\pi}{4} + D_2 \cdot (\sqrt{2})^n \sin\frac{n\pi}{4}.$$

Multiple roots are somewhat different.

Ex. 3. Find all sequences satisfying $a_{n+2} = 4(a_{n+1} - a_n)$.

Solution. Here we have $a_{n+2} - 4a_{n+1} + 4a_n = 0$. The characteristic polynomial is $r^2 - 4r + 4 = (r-2)^2$ ($r = 2$ is a double root). In this case $a_n = 2^n$ is still a solution, but the other[1] one is $a_n = n \cdot 2^n$. The general solution is consequently

$$a_n = C_1 \cdot 2^n + C_2 \cdot n2^n.$$

2. Nonhomogeneous linear equations

Let us now consider equations of the type

$$a_{n+k} + c_{k-1}a_{n+k-1} + \ldots + c_1 a_{n+1} + c_0 a_n = f(n),$$

where f depends on n, preferably as some combination of polynomials and exponential functions[2].

Proposition. The general solution of a nonhomogeneous equation is the sum of the general solution of the related homogeneous equation and a particular solution of the nonhomogeneous equation (in short GSNE=GSHE+PSNE)

By *particular* solution (PSNE) we mean just any solution, usually the one that is easy to find. Quite often the **prediction method** is used, which means that if $f(n) = W_k(n) \cdot c^n$, where W_k is a given polynomial of degree k, then we predict (except one subtle case, which will be addressed later) a solution in the similar form $a_n = V_k(n) \cdot c^n$, where V_k is also a polynomial of degree k.

Ex. 4. Find all sequences satisfying $a_{n+2} - 5a_{n+1} + 6a_n = 6 \cdot 4^n$.

Solution. GSHE is, by the previous method, $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$, and PSNE will by looked for in the form $a_n = A \cdot 4^n$ (W is here the constant 6, that is a polynomial of degree 0, so V is also predicted to be a constant). Let us now compute

$$a_{n+2} - 5a_{n+1} + 6a_n = A \cdot 4^{n+2} - 5A \cdot 4^{n+1} + 6A \cdot 4^n = 16A \cdot 4^n - 20A \cdot 4^n + 6A \cdot 4^n = 2A \cdot 4^n.$$

The above expression should be equal to $6 \cdot 4^n$, so we get $2A = 6$, so $A = 3$. Finally PSNE equals $a_n = 3 \cdot 4^n$, and the final answer is

$$\text{(GSNE)} \quad a_n = C_1 \cdot 2^n + C_2 \cdot 3^n + 3 \cdot 4^n.$$

[1] if $r = 2$ were a triple root, we would also have the third special solution $a_n = n^2 \cdot 2^n$ etc.

[2] both because we rarely encounter more complicated functions in practical problems, and because there exist so many functions that no method could conceivably cover all of them

Ex. 5. Find all sequences satisfying $a_{n+2} - 5a_{n+1} + 6a_n = 4n$.

Solution. GSHE is still $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$, and PSNE will be predicted as $a_n = An + B$ (the degree of W is 1, so V is predicted as degree 1 polynomial; the 'invisible' constant c is implicitly 1). Now

$$a_{n+2} - 5a_{n+1} + 6a_n = A(n+2) + B - 5(A(n+1) + B) + 6(An + B) = 2An - 3A + 2B.$$

We now have to find constants for which $2An - 3A + 2B = 4n$. From $2A = 4$ and $-3A + 2B = 0$ we immediately see that $A = 2, B = 3$. PSNE is therefore $a_n = 2n + 3$, and the final answer is

$$\text{(GSNE)}\ \ a_n = C_1 \cdot 2^n + C_2 \cdot 3^n + 2n + 3.$$

Remark. A common error is looking for PSNE in the form $a_n = An$, under the impression that $4n$ has no constant term. But that is wrong, as the resulting equation $2An - 3A = 4n$ cannot be solved[3].

A small subtlety appears in the below example.

Ex. 6. Find all sequences satisfying $a_{n+2} - 5a_{n+1} + 6a_n = 2^n$.

Solution. GSHE is again $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$, but let us see what would happen if we predicted $a_n = A \cdot 2^n$ (as in Ex. 4). We would compute

$$a_{n+2} - 5a_{n+1} + 6a_n = A \cdot 2^{n+2} - 5A \cdot 2^{n+1} + 6A \cdot 2^n = 4A \cdot 2^n - 10A \cdot 2^n + 6A \cdot 2^n = 0.$$

The original equation becomes now $0 = 2^n$, an obvious contradiction.

The problem is that $A \cdot 2^n$ is already a solution of the homogeneous equation, so it cannot simultaneously be a solution of the nonhomogeneous one.

The correct approach is assuming $a_n = A \cdot n \cdot 2^n$. Then

$$a_{n+2} - 5a_{n+1} + 6a_n = A \cdot (n+2) \cdot 2^{n+2} - 5A \cdot (n+1) \cdot 2^{n+1} + 6A \cdot n \cdot 2^n =$$
$$= A \cdot (4n + 8 - 10n - 10 + 6n) \cdot 2^n = -2A \cdot 2^n.$$

The resulting equation $-2A \cdot 2^n = 2^n$ is fully solvable. We obtain $A = -\frac{1}{2}$,

$$\text{(PSNE)}\ \ a_n = -\frac{1}{2} \cdot n \cdot 2^n = -n2^{n-1},$$

and finally

$$\text{(GSNE)}\ \ a_n = C_1 \cdot 2^n + C_2 \cdot 3^n - n2^{n-1}.$$

A short **explanation**: the predicted PSNE should be multiplied by n if the number c in the function $f(n) = W_k(n) \cdot c^n$ (in the present example $c = 2$) is a root of the characteristic polynomial (in the present example $r^2 - 5r + 6$), which is actually the case here. An extra explanation: the predicted PSNE should be multiplied by n^m if c is a root of the characteristic polynomial of multiplicity m.

[3] writing $A = \frac{4n}{2n-3}$ would be absurd because A has to be a constant