EIDMA. PROBLEM SET 10

1. In a theatre Alice, Brian and Chelsea can choose any three of $n+1$ seats in a particular row, but Brian wants to sit between his friends. Count the number of possible ways of doing that by two different methods and thus prove the equality

$$\sum_{k=0}^{n} k(n-k) = \binom{n+1}{3}.$$

2. How many solutions in positive integers (0 isn't positive, of course) of

$$a + b + c + d + e + f = 18$$

exist? We count $3+3+3+3+5+1$ and $3+3+3+5+1+3$ as distinct solutions. In how many of them all the numbers $a, \ldots, f$ are odd, and in how many exactly four of them are even?

3. In a group of 50 people exactly 40 can sing, 38 can play the piano and 36 can paint. Everyone has at least one of those skills, and exactly 25 people can do all three things. How many people have at least two of the three skills?

4. There are exactly $5^7 = 78125$ seven-digit numbers consisting of digits $1, 2, 3, 4, 5$. In how many of them each of $1, 2, 3, 4, 5$ appears at least once (none is absent)?

4.* Using the inclusion-exclusion principle, prove the following general formula for the number of all functions with domain $\{1, 2, \ldots, n\}$ and range (ie. the set of all values of the function) $\{1, 2, \ldots, k\}$:

$$S(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

eg. $S(7,5) = \binom{5}{0}5^7 - \binom{5}{1}4^7 + \binom{5}{2}3^7 - \binom{5}{3}2^7 + \binom{5}{4}1^7 - \binom{5}{5}0^7 = 16800.$