Introduction to Discrete Mathematics. Test #2

1. Prove that the relation R defined on the set $X = \mathbb{Z}$ by

   $mRn \equiv (2m)^2 + (3n)^2$ is divisible by 13

   is an equivalence relation. Determine its equivalence classes $[0]_R, [9]_R$. How many different equivalence classes does this relation have?

   Hints:

   reflexive: $(2m)^2 + (3m)^2 = 13m^2$
   symmetric: $(2n)^2 + (3m)^2 = 13m^2 + 13n^2 - ((2m)^2 + (3n)^2)$
   transitive: $(2m)^2 + (3p)^2 = ((2m)^2 + (3n)^2) + ((2n)^2 + (3p)^2) - 13n^2$
   $[0]_R = 13\mathbb{Z}$, $[9]_R = \{4, 9\} + 13\mathbb{Z}$, there are exactly seven equivalence classes

2. Let $X = 5\mathbb{Z}$ be the set of all integers divisible by 5, let $Y = \mathbb{Z}_+$ be the set of all positive integers. Find bijections $f : \mathbb{Z} \to X$ and $g : \mathbb{Z} \to Y$.

   (question for extra points) Find a bijection $h : \mathbb{Z} \to (X \cup Y)$.

   Possible answers: $f(n) = 5n$,

$$g(n) = \begin{cases} 2n & \text{for} \quad n > 0 \\ 1 - 2n & \text{for} \quad n \leqslant 0, \end{cases} \qquad h(n) = \begin{cases} n & \text{for} \quad n > 0 \\ 5n & \text{for} \quad n \leqslant 0. \end{cases}$$

3. Determine for what values of the parameters $a, b \in \mathbb{R}$ the operation $x \ddagger y = = 2xy - 5x + ay + b$ defined for $x, y \in \mathbb{R}$ is commutative and associative.

   Hints:

   commutative: $2xy - 5x + ay + b = 2yx - 5y + ax + b$ implies $a = -5$
   associative: for $x \ddagger y = 2xy - 5x - 5y + b$ we have
   $(0 \ddagger 0) \ddagger 1 = b \ddagger 1 = -2b - 5$
   $0 \ddagger (0 \ddagger 1) = 0 \ddagger (b - 5) = 25 - 4b$; $25 - 4b = -2b - 5$ implies $b = 15$
   Then check manually that $x \ddagger y = 2xy - 5x - 5y + 15$ is associative or use

$$2xy - 5x - 5y + 15 = \frac{(2x - 5)(2y - 5) + 5}{2}$$

   to prove that

$$(x \ddagger y) \ddagger z = \frac{(2x - 5)(2y - 5)(2z - 5) + 5}{2} = x \ddagger (y \ddagger z)$$