We assume throughout that $0 \in \mathbb{N}$.

1. Prove that the relation R defined on the set $X = \mathbb{N}$ by

$$mRn \Leftrightarrow \text{at least one of the numbers } 7m + n, 3m + 3n \text{ is divisible by } 8$$

   is an equivalence relation. Determine its equivalence classes $[0]_R, [2025]_R$.

2. Prove that the function $f : \mathbb{Z} \to \mathbb{N}$ defined as

$$f(n) = \begin{cases} n^2 - n + 1 & \text{for} \quad n \leqslant 0, \\ n^2 - n & \text{for} \quad n > 0 \end{cases}$$

   is injective but not surjective.

3. Prove that the function $g : \mathbb{Z} \to \mathbb{N}$ defined as

$$g(n) = \begin{cases} n^2 + n + 1 & \text{for} \quad n \leqslant 0, \\ n^2 + n & \text{for} \quad n > 0 \end{cases}$$

   is neither injective nor surjective.

4. Let $X = 3\mathbb{Z}$ be the set of all integers divisible by 3. Find bijections $f : \mathbb{Z} \to X$ and $g : X \to \mathbb{N}$ (and explain why they are bijections).

   (question for extra points) Find a bijection $h : (X \cup \mathbb{N}) \to (\mathbb{Z} \setminus X)$.

5. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2 + 2x + 2$. Find the least (smallest) and greatest (largest) elements of the sets:

$$Y = \{y \in \mathbb{R} : (\exists x \in \mathbb{R})\ (y = f(x) \wedge x^2 < 4)\},$$

$$Z = \{z \in \mathbb{R} : (\exists x \in \mathbb{R}) f(x) \cdot z = 1\},$$

   or explain why those elements do not exist.