Przyklad 1d. Obliczyc $\int\int\int_V (x+z)\mathrm{dxdydz}$ **gdzie V - bryla ograniczona stozkiem** $z=\sqrt{x^2+4y^2}$ **i plaszczyzna** $z=2$.

$V=\{(x,y,z)\}:\sqrt{x^2+4y^2}\le z\le 2,\ x^2+4y^2\le 4\}.$

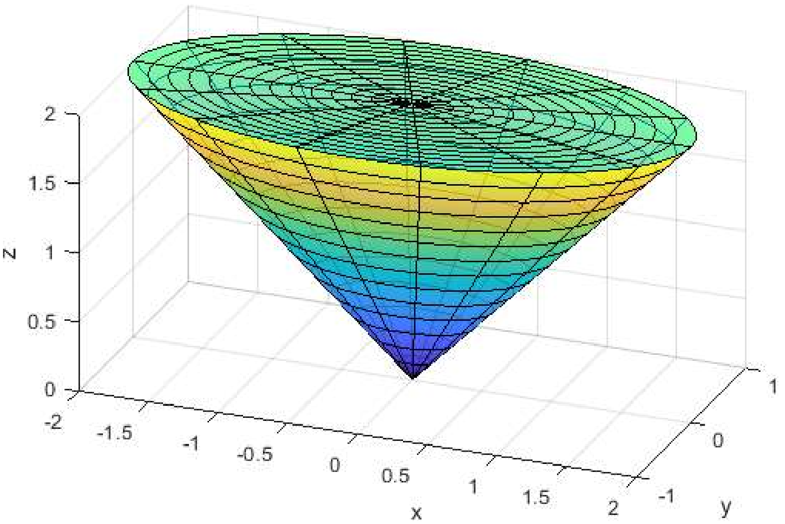
Wykorzystamy wspolrzedne walcowo-eliptyczne:

$x=2\,r\cos t,\ y=r\sin t,\ z=z,\ J=2r, r\in[0,1], t\in[0,2\pi], z\in[2r,2]$

Obliczamy

$$\int_0^{2\pi}\left(\int_0^1\left(\int_{2r}^2(2r\cos t+z)2r\mathrm{dz}\right)\mathrm{dr}\right)\mathrm{dt}=\int_0^{2\pi}\cos t\,\mathrm{dt}\cdot\int_0^1\left(\int_{2r}^2(2r)2r\mathrm{dz}\right)\mathrm{dr}+\int_0^{2\pi}\mathrm{dt}\cdot\int_0^1\left(\int_{2r}^2(z)2r\mathrm{dz}\right)\mathrm{dr}=$$

$$=0+2\pi\cdot\int_0^1 2r\left(\frac{z^2}{2}\Big|_{2r}^2\right)\mathrm{dr}=2\pi\cdot\int_0^1 r(4-4r^2)\mathrm{dr}=2\pi\cdot 1=2\pi$$



Przyklad 1e. Obliczyc

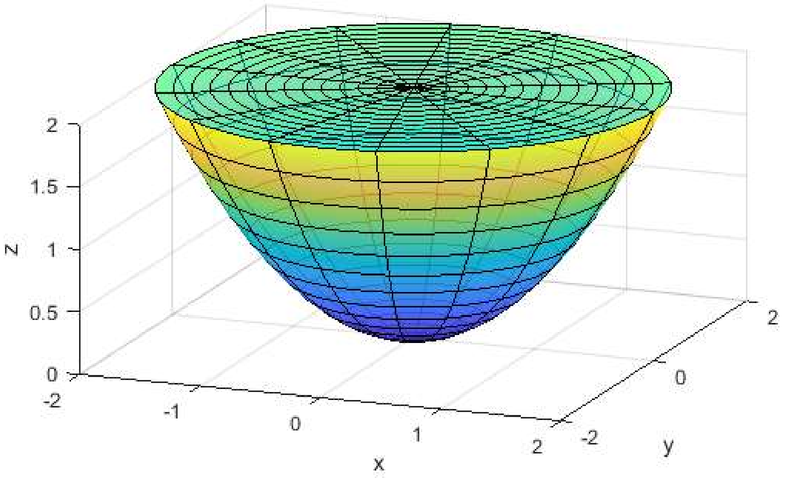
$\int\int\int_V (1+x^2+y^2)\mathrm{dxdydz},\ V=\left\{(x,y,z):\frac{1}{2}(x^2+y^2)\le z\le 2\right\}.$

Wykorzystamy wspolrzedne walcowe:

$x=r\cos t,\ y=r\sin t,\ z=z,\ J=r, r\in[0,1], t\in[0,2\pi],\ r\in[0,2],\ z\in\left[\frac{1}{2}r^2,2\right]$

Obliczamy $\int_0^{2\pi}\left(\int_0^2\left(\int_{\frac{1}{2}r^2}^2(1+r^2)r\mathrm{dz}\right)\mathrm{dr}\right)\mathrm{dt}=\int_0^{2\pi}\mathrm{dt}\cdot\int_0^2\left(\int_{\frac{1}{2}r^2}^2(r+r^3)\mathrm{dz}\right)\mathrm{dr}=$

$$=2\pi\cdot\int_0^2(r+r^3)\left(2-\frac{1}{2}r^2\right)\mathrm{dr}=2\pi\cdot\int_0^2\left(2r+2r^3-\frac{1}{2}r^3-\frac{1}{2}r^5\right)\mathrm{dr}=2\pi\cdot\frac{14}{3}=\frac{28}{3}\pi$$



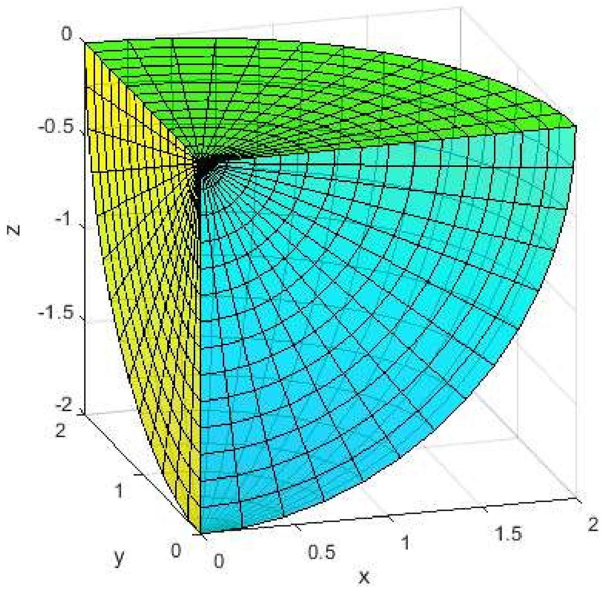
Przyklad 1f. Obliczyc calke $\int\int\int_V z\sqrt{x^2+y^2}\,\mathrm{dxdydz}$ **po oktancie (jedna osma) kuli** $V=\{(x,y,z)\}: x^2+y^2+z^2\le 4,\ x\ge 0,\ y\ge 0,\ z\le 0\}.$

Wykorzystamy zmienne sferyczne $x=r\cdot\cos t\cdot\cos u, y=r\cdot\cos t\cdot\sin u, z=r\cdot\sin t$**, wtedy** $\sqrt{x^2+y^2+z^2}=r,\ J=r^2\cos t, U:\ \ r\in[0;2],\ t\in\left[-\frac{\pi}{2},0\right],\ \ u\in\left[0;\frac{\pi}{2}\right].$

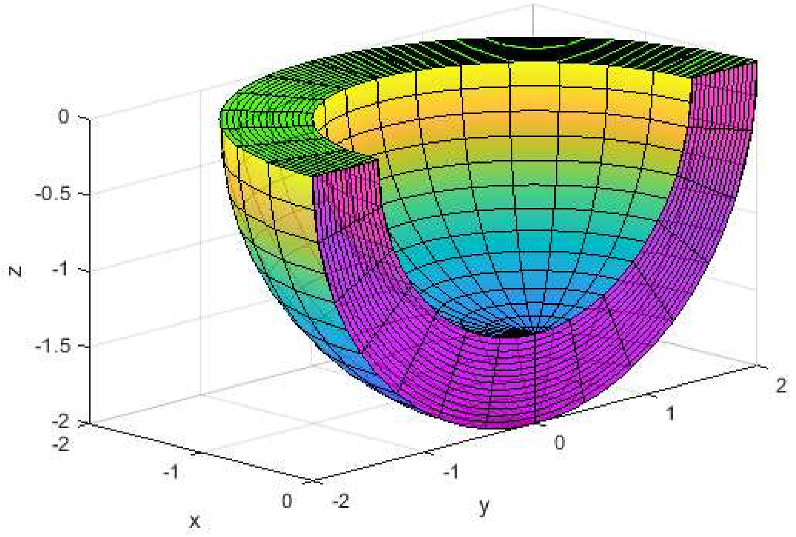
Obliczymy

$$\int_0^{\frac{\pi}{2}}\left(\int_{-\frac{\pi}{2}}^0\left(\int_0^2 r\sin t\cdot\sqrt{r^2\cos^2 t}\cdot r^2\cos t\,\mathrm{dr}\right)\mathrm{dt}\right)\mathrm{du}=\int_0^{\frac{\pi}{2}}\left(\int_{-\frac{\pi}{2}}^0\left(\int_0^2 r\sin t\cdot r\,\mathrm{cost}\cdot r^2\cos t\,\mathrm{dr}\right)\mathrm{dt}\right)\mathrm{du}=$$

$$\int_0^{\frac{\pi}{2}}\left(\int_{-\frac{\pi}{2}}^0\left(\int_0^2 r^4\sin t\cos^2 t\,\mathrm{dr}\right)\mathrm{dt}\right)\mathrm{du}=\int_0^{\frac{\pi}{2}}\mathrm{du}\cdot\int_0^2 r^4\ \mathrm{dr}\cdot\int_{-\frac{\pi}{2}}^0\sin t\cos^2 t\,\mathrm{dt}=\frac{\pi}{2}\cdot\frac{32}{5}\cdot\left(-\frac{1}{3}\right)=-\frac{16\pi}{15}$$



Przyklad 1g. Obliczyc $\int\int\int_V \frac{2z}{x^2+y^2+z^2+4}\mathrm{dxdydz},\ V:\ x\le 0, z\le 0,\ 2\le x^2+y^2+z^2\le 4.$

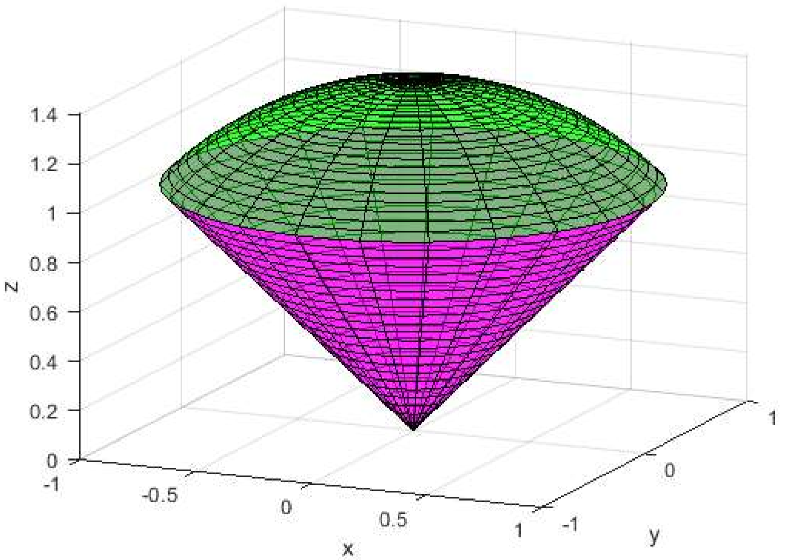


Wspolrzedne sferyczne $x = r \cdot \cos t \cdot \cos \theta, y = r \cdot \cos t \cdot \sin \theta, z = r \cdot \sin t$**, wtedy** $\sqrt{x^2 + y^2 + z^2} = r,\ J = r^2 \cos t,\ U:\ \ r \in [\sqrt{2}\,; 2],\ t \in \left[-\frac{\pi}{2}, 0\right],\ \ \theta \in \left[\frac{\pi}{2}; \frac{3\pi}{2}\right].$

$$\int\int\int_V \frac{2z}{x^2 + y^2 + z^2 + 4} \mathrm{dxdydz} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\int_{-\frac{\pi}{2}}^{0} \left(\int_{\sqrt{2}}^{2} \frac{2r \sin t}{r^2 + 4} r^2 \cos t \,\mathrm{dr} \right) \mathrm{dt} \right) d\theta =$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta \cdot \int_{-\frac{\pi}{2}}^{0} \sin t \cos t \,\mathrm{dt} \cdot \int_{\sqrt{2}}^{2} \frac{2r^3 \mathrm{dr}}{r^2 + 4} = \pi \cdot \left(-\frac{1}{2}\right) \cdot \left(2 - 4\ln\frac{4}{3}\right) = \pi \cdot \left(\ln\frac{16}{9} - 1\right).$$

Przyklad 2a. Obliczyc objetosc bryly wycietej z kuli $x^2 + y^2 + z^2 \le 2$ **przez stozek** $z \ge \sqrt{x^2 + y^2}$**.**



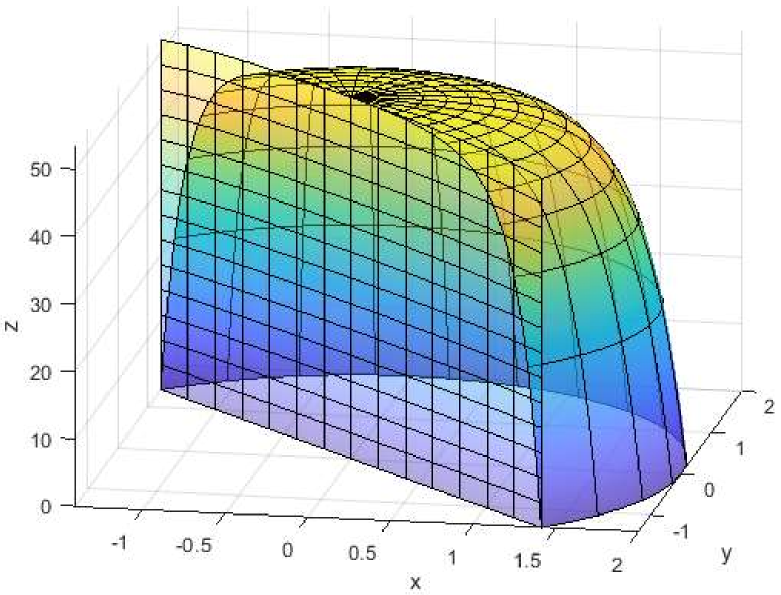
Wykorzystamywspolrzedne sferyczne:

$$x = r \cdot \cos\theta \cdot \cos\phi,\ y = r \cdot \sin\theta \cdot \cos\phi,\ z = r \cdot \sin\phi,\ J = r^2 \cdot \cos\phi,\ r \in [0; \sqrt{2}], \phi \in \left[\frac{\pi}{4}; \frac{\pi}{2}\right], \theta \in [0; 2\pi].$$

$$V=\int\int\int_V \mathrm{dxdydz}=\int_0^{2\pi}\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^{\sqrt{2}} r^2\cdot\cos\phi\,\mathrm{dr}\right)d\phi\right)d\theta=\int_0^{2\pi}d\theta\cdot\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos\phi\,d\phi\cdot\int_0^{\sqrt{2}} r^2\mathrm{dr}=$$

$$=2\pi\cdot\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{2\sqrt{2}}{3}=\frac{4}{3}\pi(\sqrt{2}-1).$$

Przyklad 2b. Obliczyc objetosc bryly $V=\left\{(x,y,y):0\le z\le e^4-e^{x^2+y^2},\ x+y\ge 0\right\}.$



Wspolrzedne walcowe:

$x=r\cos\phi,\ y=r\sin\phi, z=z,\ J=r,\ r\in[0,2], \phi\in\left[-\frac{\pi}{4},\frac{3\pi}{4}\right], z\in\left[0,e^4-e^{r^2}\right].$

$$|V|=\int\int\int_V 1\mathrm{dxdydz}=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\int_0^2\left(\int_0^{e^4-e^{r^2}} r\mathrm{dz}\right)\mathrm{dr}\right)d\phi=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}d\phi\cdot\int_0^2 r\left(e^4-e^{r^2}\right)\mathrm{dr}=\pi\cdot\int_0^2 r\,e^4-re^{r^2}\mathrm{dr}=$$

$$=\pi\cdot\left[\frac{e^4r^2}{2}-\frac{e^{r^2}}{2}\right]_0^2=\frac{\pi}{2}(3e^4+1).$$