ON LATTICES CONNECTED WITH VARIOUS TYPES OF CLASSES OF ALGEBRAIC STRUCTURES

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Abstract. This survey paper reviews some recent results related to various derived lattices connected with various types of classes of algebraic structures which were obtained by the authors.

1. Introduction

This survey paper presents recent results obtained for lattices of subclasses of certain types. Mainly, we focus on representing lattices by lattices of relatively axiomatizable classes and those of [finitary] prevarieties, also mentioning some general algebraic and computational properties of those lattices.

Study of such lattices has a long history and goes back to G. Birkhoff and A. I. Maltsev. In [6] and [16], they have independently asked about which lattices can be represented as lattices of [quasi]varieties; that is, classes defined by [quasi-]identities. It is one of the oldest and hardest problems in lattice theory. A number of remarkable results was obtained concerning this question of Birkhoff and Maltsev. An advance in the Birkhoff-Maltsev problem was made by K. V. Adaricheva, W. Dziobiak and V. A. Gorbunov by describing algebraic atomistic lattices isomorphic to quasivariety lattices in [2], see also V. A. Gorbunov [11, Theorem 5.3.17]. It is also known (V. A. Gorbunov [11]) that all atomistic algebraic quasivariety lattices are isomorphic to so-called lattices of algebraic subsets of algebraic lattices. We also note that those lattices are dual to lattices of suitable first-order theories, cf. results of K. Adaricheva, J. B. Nation [3] and [18] and also the talk of G. F. McNulty on lattices of equational theories [17]. For other results concerning this topic, we refer to the book of V. A. Gorbunov [11, Chapter 5], see also the survey paper by M. Adams, K. Adaricheva, W. Dziobiak, and A. Kravchenko [1], as well as to bibliography lists in those two. Besides that, lattices of pseudovarieties of finite algebras were investigated in a number of papers, see, for example, P. Agliano and J. B. Nation [4].

A. M. Nurakunov proved in [21] that there are quasivarieties of algebras (structures with no relation in the signature) such that the set of finite sublattices of their quasivariety lattices is not computable, see Section 7. This result shows, in particular, that finding a complete description of quasivariety lattices should be very hard. But there are some restricted versions of the Birkhoff-Maltsev problem still of big interest.

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While sub[quasi]variety lattices were studied in a considerable extent, lattices of other first-order axiomatizable classes remain almost not touched. In [22], D. E. Pal'chunov has proved that any at most countable complete lattice is isomorphic to a lattice of relatively axiomatizable classes. In [22, Problem 1], he asked whether the same result holds for an *arbitrary* complete lattice. We answer the latter question in the positive in Theorem 5.2, which is based on a result of V. A. Gorbunov [9].

All classes are *abstract*; that is, they are closed under isomorphic copies. For example, when writing $\{\mathcal{A}_i \mid i \in I\}$ for a set I, we always mean the class of isomorphic copies of structures from the set $\{\mathcal{A}_i \mid i \in I\}$.

For all the concepts which are not defined here, we refer to V. A. Gorbunov [11].

2. Basic concepts

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ. Let also $\mathbf{T}(\sigma)$ denote the variety of σ-structures defined by the identity $\forall xy\ x = y$.

Following V. A. Gorbunov [11], for a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, let $\mathbf{V}(\mathbf{K})$ [$\mathbf{Q}(\mathbf{K})$, respectively] denote the least [quasi-]variety containing $\mathbf{K}$. Let $\mathbf{H}(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ which are homomorphic images of structures from $\mathbf{K}$; let $\mathbf{P}(\mathbf{K})$ [$\mathbf{P}^\omega(\mathbf{K})$, respectively] denote the class of structures from $\mathbf{K}(\sigma)$ which are isomorphic to Cartesian products of [finitely many] structures from $\mathbf{K}$; let $\mathbf{P}_s(\mathbf{K})$ [$\mathbf{P}_s^\omega(\mathbf{K})$, respectively] denote the class of structures from $\mathbf{K}(\sigma)$ which are isomorphic to subdirect products of [finitely many] structures from $\mathbf{K}$; let $\mathbf{L}_s(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ which are isomorphic to superdirect limits of structures from $\mathbf{K}$; and let $\mathbf{S}(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ which are isomorphic to substructures of structures from $\mathbf{K}$. Finally, let $\mathbf{K}_{fin}$ denote the class of finite members of $\mathbf{K}$.

According to Birkhoff's Theorem (see V. A. Gorbunov [11, Section 2.3]),

$$\mathbf{V}(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}) = \mathbf{HP}_s\mathbf{S}(\mathbf{K}) = \mathbf{HP}_s(\mathbf{K}),$$

while according to V. A. Gorbunov and V. I. Tumanov [14, Theorem 5.2] (see also V. A. Gorbunov [11, Theorem 2.3.6]),

$$\mathbf{Q}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s\mathbf{S}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{K}).$$

A class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is a [*finitary*] *prevariety*, if $\mathbf{K} = \mathbf{SP}(\mathbf{K}) = \mathbf{P}_s\mathbf{S}(\mathbf{K})$ [$\mathbf{K} = \mathbf{SP}^\omega(\mathbf{K}) = \mathbf{P}_s^\omega\mathbf{S}(\mathbf{K})$, respectively]. The notion of a finitary prevariety (in case of signature containing no relation symbols) was introduced by A. Vernitski in [26]. According to B. Banaschewski and H. Herrlich [5], a class is a prevariety if and only if it can be defined by infinite implications.

Definition 2.1. [11, Section 2.5] Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is $\mathbf{K}$-[*quasi-*]*equational*, if $\mathbf{K}' = \mathbf{K} \cap \operatorname{Mod}(\Sigma)$ for some set Σ of [quasi-]identities of signature σ.

For the following concept, see V. A. Gorbunov [11] and also [25].

Definition 2.2. Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is a [*finitary*] $\mathbf{K}$-*prevariety*, if $\mathbf{K}' = \mathbf{K} \cap \mathbf{A}$ for some [finitary] prevariety $\mathbf{A} \subseteq \mathbf{K}(\sigma)$; $\mathbf{K}'$ is a $\mathbf{K}$-[*quasi*]*variety*, if $\mathbf{K}' = \mathbf{K} \cap \mathbf{A}$ for some [quasi]variety $\mathbf{A} \subseteq \mathbf{K}(\sigma)$.

Equivalently, $\mathbf{K}'$ is a [finitary] $\mathbf{K}$-prevariety if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{SP}(\mathbf{K}')$ [$\mathbf{K}' = \mathbf{K} \cap \mathbf{SP}^\omega(\mathbf{K}')$, respectively]. Similarly, $\mathbf{K}'$ is a $\mathbf{K}$-[quasi]variety if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{V}(\mathbf{K}')$ [$\mathbf{K}' = \mathbf{K} \cap \mathbf{Q}(\mathbf{K}')$, respectively].

Definition 2.3. A class $\mathbf{K} \subseteq \mathbf{K}(\sigma)_{fin}$ is a *pseudo-quasivariety*, if it is a finitary prevariety.

Note that $\mathbf{K} \subseteq \mathbf{K}(\sigma)_{fin}$ is a pseudo-quasivariety if and only if it is a [finitary] $\mathbf{K}_{fin}$-prevariety, if and only if it is a $\mathbf{K}_{fin}$-quasivariety.

Let $\mathrm{Lv}(\mathbf{K})$ denote the set of all $\mathbf{K}$-equational subclasses of $\mathbf{K}$, while $\mathrm{Lq}(\mathbf{K})$ denotes the set of all $\mathbf{K}$-quasi-equational subclasses of $\mathbf{K}$. Let also $\mathrm{Lp}(\mathbf{K})$ [$\mathrm{Lp}^\omega(\mathbf{K})$, respectively] denote the set of all [finitary] $\mathbf{K}$-prevarieties. Ordered with respect to set inclusion, all the three form complete lattices. Note that in the case of [finitary] prevarieties, we also allow the case when the ground of a lattice is a proper class.

Definition 2.4. Let L be a complete lattice. A subset $A \subseteq L$ is a *complete meet subsemilattice* of L, if $\bigwedge X \in A$ for any $X \subseteq A$. A complete meet subsemilattice $A \subseteq L$ is an *algebraic subset* of L, if $\bigvee X \in A$ for any non-empty up-directed subset X of A.

A binary relation R on a meet semilattice $\langle S, \wedge\rangle$ is *distributive*, if for any $a, b, c \in S$ relation $(c, a \wedge b) \in R$ implies that $c = a' \wedge b'$ for some $a', b' \in S$ such that $(a', a) \in R$ and $(b', b) \in R$. The equality relation $=$ is obviously distributive.

For a meet semilattice $\langle S, \wedge, 1\rangle$ with unit and for any binary relation $R \subseteq S^2$, let $\mathrm{Sub}(S, R)$ denote the set of all *R-closed subsemilattices* of S; that is, $X \in \mathrm{Sub}(S, R)$ if and only if the following conditions hold:

- $\bigwedge F \in X$ for all finite $F \subseteq X$;
- $b \in X$ and $(a, b) \in R$ imply $a \in X$.

For a complete lattice L, let $\mathrm{Sub}_c(L, R)$ denote the set of all *complete R-closed meet subsemilattices* of L, while $\mathrm{Sp}(L, R)$ denotes the set of all *algebraic subsets* of L which are R-closed. Let also $\mathrm{F}(L, R)$ denote the set of R-closed *filters* of L. We write $\mathrm{Sub}(L)$, $\mathrm{Sub}_c(L)$, $\mathrm{Sp}(L)$, and $\mathrm{F}(L)$ instead of $\mathrm{Sub}(L, =)$, $\mathrm{Sub}_c(L, =)$, $\mathrm{Sp}(L, =)$, and $\mathrm{F}(L, =)$, respectively. Ordered by inclusion, $\mathrm{Sub}(L, R)$, $\mathrm{Sub}_c(L, R)$, $\mathrm{Sp}(L, R)$ form complete lattices, while ordered by reverse inclusion, $\mathrm{F}(L, R)$ also forms a complete lattice.

3. Representing by congruence lattices

For a structure $\mathcal{A} \in \mathbf{K}(\sigma)$ and for a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, let $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A}$ denote the set of congruences θ on $\mathcal{A}$ such that $\mathcal{A}/\theta \in \mathbf{K}$. If $\mathbf{K} = \mathbf{K}(\sigma)$, then we write $\mathrm{Con}\,\mathcal{A}$ instead of $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A}$. For $\theta, \theta' \in \mathrm{Con}\,\mathcal{A}$, we write $\theta' \,\mathrm{E}\, \theta$, if $\mathcal{A}/\theta'$ embeds into $\mathcal{A}/\theta$. Then E is called the *embedding relation*. Obviously, this relation is distributive.

The next theorem combines the characterization theorem proved for quasivarieties by V. A. Gorbunov and V. I. Tumanov [13, 14], see also V. A. Gorbunov [11, Corollaries 5.2.2, 5.2.6] with its analogue for [finitary] prevarieties obtained in [25].

Theorem 3.1. *Let $\mathbf{A} \subseteq \mathbf{K}(\sigma)$ be a prevariety and let $\mathcal{A} \in \mathbf{A}$. The following holds:*

$$\mathrm{Lp}(\mathbf{H}(\mathcal{A}) \cap \mathbf{A}) \cong \mathrm{Sub}_c(\mathrm{Con}_{\mathbf{A}}\,\mathcal{A}, \mathrm{E});$$
$$\mathrm{Lp}^\omega(\mathbf{H}(\mathcal{A}) \cap \mathbf{A}) \cong \mathrm{Sub}(\mathrm{Con}_{\mathbf{A}}\,\mathcal{A}, \mathrm{E}).$$

If $\mathcal{A}$ is $[l]$-projective in $\mathbf{A}$, then

$$\mathrm{Lq}(\mathbf{H}(\mathcal{A}) \cap \mathbf{A}) \cong \mathrm{Sp}(\mathrm{Con}_{\mathbf{A}}\,\mathcal{A}, \mathrm{E});$$
$$\mathrm{Lv}(\mathbf{H}(\mathcal{A}) \cap \mathbf{A}) \cong \mathrm{F}(\mathrm{Con}_{\mathbf{A}}\,\mathcal{A}, \mathrm{E}).$$

In particular, one gets the following

Corollary 3.2. [11, Corollaries 5.2.2, 5.2.5] *Let $\mathbf{A} \subseteq \mathbf{K}(\sigma)$ be a prevariety and let $\mathcal{F}_{\mathbf{K}}(\omega) \in \mathbf{A}$ be a $\mathbf{K}$-free structure of countable rank. The following holds:*

$$\mathrm{Lq}(\mathbf{A}) \cong \mathrm{Sp}\big(\mathrm{Con}_{\mathbf{K}}\,\mathcal{F}_{\mathbf{K}}(\omega), \mathrm{E}\big);$$
$$\mathrm{Lv}(\mathbf{A}) \cong \mathrm{F}\big(\mathrm{Con}_{\mathbf{K}}\,\mathcal{F}_{\mathbf{K}}(\omega), \mathrm{E}\big).$$

For any class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ and any cardinal κ, let $\mathbf{K}_\kappa$ denote the class of κ-generated structures from $\mathbf{K}$. The following statement is an analogue of Corollary 3.2 for prevarieties.

Corollary 3.3. [25] *For any prevariety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ and for any cardinal κ,*

$$\mathrm{Lp}(\mathbf{K}_\kappa) \cong \mathrm{Sub}_c\big(\mathrm{Con}_{\mathbf{K}}\,\mathcal{F}_{\mathbf{K}}(\kappa), \mathrm{E}\big).$$

We note that if $\mathbf{K}$ is a prevariety, then for any structure $\mathcal{A}$, the congruence lattice $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A}$ is a complete lattice, which is algebraic if and only if $\mathbf{K}$ is a quasivariety. In the next section, we will state a partial converse of Corollary 3.2. More precisely, any complete lattice is isomorphic to the lattice of relative varieties of a prevariety, any lattice of algebraic subsets of an algebraic lattice is isomorphic to a quasivariety lattice, any lattice of complete subsemilattices of a complete lattice is isomorphic to a prevariety lattice, and any subsemilattice lattice is isomorphic to a finitary prevariety lattice, see Propositions 4.2, 4.3, and 4.4.

A well-known and long standing problem in lattice theory asks whether any finite lattice is isomorphic to the congruence lattice of a finite algebra of finite signature. The next result proved by A. M. Nurakunov [19] shows that any finite lattice is isomorphic to a *relative* congruence lattice of a finite algebra of finite signature.

Theorem 3.4. [19] *For any finite lattice L, there is a quasivariety $\mathbf{K}$ of unars [pointed Abelian groups, respectively] and a finite algebra $\mathcal{A} \in \mathbf{K}$ such that $L \cong \mathrm{Con}_{\mathbf{K}}(\mathcal{A})$.*

The following result obtained by A. M. Nurakunov [20] gives a description of lattices of subvarieties in terms of congruence lattices.

Theorem 3.5. [20] *A lattice is isomorphic to a variety lattice if and only if it is dually isomorphic to the congruence lattice of a monoid with two additional unary operations possessing certain properties.*

Based on ideas from [20], K. Adaricheva and J. B. Nation proved in [3] an analogue of Theorem 3.5 for quasivariety lattices: Quasivariety lattices are exactly lattices dually isomorphic to congruence lattices of semilattices endowed with unary operations possessing certain properties. Besides that, J. B. Nation proved in [18, Corollary 16] that the congruence lattice of any semilattice with operators is dually isomorphic to the lattice of subprevarieties of a prevariety.

4. Representation by lattices of subclasses

4.1. **Relation symbols.** Let $\sigma = \{p_i \mid i \in I\}$ be a signature consisting of unary relation symbols only. Furthermore, for any set $X \subseteq I$, let $\mathcal{A}_X$ denote a structure from $\mathbf{T}(\sigma)$ such that $\mathcal{A}_X \models \forall x\, p_i(x)$ iff $i \in X$. Obviously, $\mathbf{T}(\sigma)$ consists of isomorphic copies of structures $\mathcal{A}_X$, $X \subseteq I$.

Let $\langle X, C\rangle$ be a closure space and $\mathbb{L}(X, C)$ be the closure lattice on X. We put

$$\sigma(X) = \{p_x \mid x \in X\}.$$

Let $\Sigma(X,C)$ consist of (in general infinite) implications of the form

$$\forall x \bigwedge_{a\in A} p_a(x) \to p_b(x), \quad A \subseteq X,\ b \in C(A).$$

Of course, if the set X is finite, then the signature $\sigma(X)$ is finite, while $\Sigma(X,C)$ becomes a finite set of quasi-identities.

The class $\mathrm{Mod}\big(\Sigma(X,C)\big)$ is obviously closed under substructures and Cartesian products, whence it is a prevariety. Therefore, the class $\mathbf{K}(X,C) = \mathrm{Mod}\big(\Sigma(X,C)\big) \cap \mathbf{T}\big(\sigma(X)\big)$ is also a prevariety.

Lemma 4.1. [25] *For any closure space $\langle X, C\rangle$, the class $\mathbf{K}(X,C)$ consists of isomorphic copies of structures $\mathcal{A}_B$, where $B \in \mathbb{L}(X,C)$.*

The following proposition shows, in particular, that any complete lattice is isomorphic to the lattice of relative equational classes of a prevariety. Originally, it was proved by V. A. Gorbunov [9, Example 4.9]. In [25] M. Semenova and A. Zamojska-Dzienio [25] gave a short direct proof; a sketch of it is presented below.

Proposition 4.2. *For any complete lattice L, there are a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L^\partial \cong \mathrm{Lv}\big(\mathbf{K}\big)$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}\big(\mathbf{K}\big)$.*

Sketch of proof. Since the lattice L is complete there is a closure space $\langle X, C\rangle$ such that $L \cong \mathbb{L}(X,C)$. Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{K}(X,C)$. Then $\mathbf{K}$ is a prevariety and a map $\varphi\colon \mathbb{L}(X,C) \to \mathrm{Lv}(\mathbf{K})$ defined by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_F \in \mathbf{T}(\sigma) \mid F \in \mathbb{L}(X,C) \text{ and } B \subseteq F\}, \quad B \in \mathbb{L}(X,C),$$

establishes a dual lattice isomorphism. □

The following proposition is a finitary analogue of Proposition 4.2 for prevarieties.

Proposition 4.3. [25] *For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting only of unary relation symbols and a finitary prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sub}(S) \cong \mathrm{Lp}^\omega(\mathbf{K})$.*

Combining Propositions 4.2-4.3, one gets the following proposition. A part of this result concerning relative [quasi]variety lattices was proved by V. A. Gorbunov and V. I. Tumanov [13, 14], see also V. A. Gorbunov [11, Theorem 5.2.8]. In the present form, it was proved in [25].

Proposition 4.4. *For any complete algebraic lattice L, there are a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L^\partial \cong \mathrm{Lv}\big(\mathbf{K}\big)$, $\mathrm{Sp}(L) \cong \mathrm{Lq}\big(\mathbf{K}\big)$, $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K})$, and $\mathrm{Sub}(L) \cong \mathrm{Lp}^\omega(\mathbf{K})$.*

From Proposition 4.4, we get also the following statement which appeared in [25].

Corollary 4.5. *The class of complete dually algebraic lattices coincides with the class of lattices of relative equational classes of quasivarieties.*

Proposition 4.6. *For any complete upper continuous lattice L, there is a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sp}(L)$ embeds into $\mathrm{Lq}\big(\mathbf{K}\big)$.*

In general, for a complete upper continuous lattice L, the lattice $\mathrm{Sp}(L)$ is not necessarily *isomorphic* to $\mathrm{Lq}(\mathbf{K})$, see [25]. However, it is the case when L is algebraic, as Proposition 4.4 above shows.

Remark 4.7. It is well-known that quasivariety lattices are completely join-semidistributive and dually algebraic, cf. V. A. Gorbunov [11, Theorem 5.1.12 and Proposition 5.1.1]. In contrast, examples given in [25] show that, in general, lattices of the form $\mathrm{Lq}(\mathbf{K})$ and $\mathrm{Lp}(\mathbf{K})$, where $\mathbf{K}$ is a prevariety, are neither join-semidistributive nor even lower continuous.

Corollary 4.8. [25] *There are prevarieties* $\mathbf{K}$ *such that neither* $\mathrm{Lq}(\mathbf{K})$ *nor* $\mathrm{Lp}(\mathbf{K})$ *embed into a quasivariety lattice.*

Using similar methods one can also prove that any complete lattice is isomorphic to the lattice of relative equational classes of a class of signature with one unary relation symbol and constant symbols as well as of signature containing only constant symbols.

4.2. **A relation symbol and constants.** Let $\langle X, C\rangle$ be a fixed closure space. We consider the signature $\sigma_p(X) = \{p\} \cup \{c_x \mid x \in X\}$, where p is a unary relation symbol and c_x is a constant symbol for any $x \in X$.

Let $\mathbf{K}' \subseteq \mathbf{K}\big(\sigma_p(X)\big)$ be the class of structures $\mathcal{A} = \langle A; \sigma_p(X)\rangle$ such that for any $a \in A$, there is $x \in X$ with $a = c_x^{\mathcal{A}}$, and satisfying the following first-order sentences

$$\begin{aligned} &\forall xy\ c_u = c_v \to x = y, && u \neq v \text{ in } X;\\ &\forall x\ c_u = c_v \to p(x), && u \neq v \text{ in } X;\\ &\forall xy \bigwedge_{x\in X} p(c_x) \to x = y. \end{aligned}$$

Furthermore, for any set $U \subseteq X$, let $\mathcal{P}_U$ denote a structure from $\mathbf{K}'$ such that $\mathcal{P}_U \models p(c_x)$ iff $x \in U$. Obviously, $\mathbf{K}'$ consists of isomorphic copies of structures $\mathcal{P}_U$, $U \subseteq X$. Moreover, $\mathcal{P}_X$ is the trivial structure.

Lemma 4.9. *The following statements hold for any set* X.

(i) *If* $A, B \subseteq X$ *then* $\mathcal{P}_A \in \mathbf{H}(\mathcal{P}_B)$ *if and only if* $B \subseteq A$.
(ii) *Let* $\{A_i \mid i \in I\} \subseteq X$ *and let* $A \subseteq X$. *Then the structure* $\mathcal{A} = \mathcal{P}_A \in \mathbf{K}'$ *is isomorphic to a substructure in* $\mathcal{B} = \prod_{i\in I} \mathcal{P}_{A_i}$ *if and only if* $A = \bigcap_{i\in I} A_i$.

Let $\Sigma_p(X, C)$ consist of the following (in general infinite) implications of the form

$$\bigwedge_{u\in U} p(c_u) \to p(c_v), \qquad U \subseteq X,\ v \in C(U).$$

Of course, if the set X is finite, then the signature $\sigma_p(X)$ is finite, while $\Sigma_p(X, C)$ becomes a finite set of quasi-identities. Let $\mathbf{K}_p(X, C) = \mathbf{K}' \cap \mathrm{Mod}\big(\Sigma_p(X, C)\big)$.

Lemma 4.10. *For any closure space* $\langle X, C\rangle$, *the class* $\mathbf{K}_p(X, C)$ *consists of isomorphic copies of structures* $\mathcal{P}_B$, *where* $B \in \mathbb{L}(X, C)$.

Proposition 4.11. *For any complete lattice* L, *there is a signature* σ *consisting of one unary relation symbol and* $|L|$ *many constant symbols and there is a class* $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ *such that* $L \cong \mathrm{Lv}\big(\mathbf{K}\big)$ *and* $\mathrm{Sub}_c(L^\partial) \cong \mathrm{Lp}\big(\mathbf{K}\big)$.

Sketch of proof. Since the lattice L is complete, there is a closure space $\langle X, C\rangle$ such that $L^\partial \cong \mathbb{L}(X, C)$. Let $\sigma = \sigma_p(X)$ and let $\mathbf{K} = \mathbf{K}_p(X, C)$. It follows from Lemma 4.1 that the class $\mathbf{K}$ consists of isomorphic copies of structures $\mathcal{P}_{\psi(a)}$, where $a \in L$. Now, the map $\varphi\colon \mathbb{L}(X, C) \to \mathrm{Lv}(\mathbf{K})$ defined by the rule

$$\varphi\colon B \mapsto \{\mathcal{P}_F \in \mathbf{K}' \mid B \subseteq F \in \mathbb{L}(X, C)\}, \quad B \in \mathbb{L}(X, C),$$

establishes a dual isomorphism. Moreover, the map $\varphi'\colon \mathrm{Sub}_c(L^\partial) \to \mathrm{Lp}(\mathbf{K})$ defined by the rule

$$\varphi'\colon B \mapsto \{\mathcal{P}_{\psi(b)} \in \mathbf{K}' \mid b \in B\}, \quad B \in \mathrm{Sub}_c(L^\partial),$$

is a lattice isomorphism. □

Proposition 4.12. *For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting of one unary relation symbol and $|S|$ many constant symbols and there is a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ such that* $\mathrm{Sub}(S) \cong \mathrm{Lp}^\omega(\mathbf{K})$.

Sketch of proof. Let $\sigma = \{p\} \cup \{c_x \mid x \in S\}$ consist of a unary relation symbol p and constant symbols c_x, $x \in S$, and let the class $\mathbf{K}$ consist of isomorphic copies of structures $\mathcal{P}_{\downarrow a}$, where $a \in S$. Define a map $\varphi\colon \mathrm{Sub}(S) \to \mathrm{Lp}^\omega(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{P}_{\downarrow b} \in \mathbf{K} \mid b \in B\}, \quad B \in \mathrm{Sub}(S).$$

It is a lattice isomorphism. □

4.3. **Only constants.** Let $\langle X, C\rangle$ be a fixed closure space. We consider the signature $\sigma(X) = \{c\} \cup \{c_x \mid x \in X\}$, where c_x is a constant symbol for any $x \in X$ as well as c is a constant symbol. In fact, one can proceed without this additional constant c, but it is just more convenient to have it.

Let $\mathbf{K}' \subseteq \mathbf{K}\big(\sigma(X)\big)$ be the class of structures $\mathcal{A} = \langle A; \sigma(X)\rangle$ such that for any $a \in A$, $a = c^{\mathcal{A}}$ or there is $x \in X$ with $a = c_x^{\mathcal{A}}$ and satisfying the following first-order sentences

$$\forall xy\; c_u = c_v \to c_u = c, \quad u \neq v \text{ in } X.$$

Furthermore, for any set $U \subseteq X$, let $\mathcal{F}_U$ denote a structure from $\mathbf{K}'$ such that $\mathcal{F}_U \models c_x = c$ iff $x \in U$. Obviously, $\mathbf{K}'$ consists of isomorphic copies of structures $\mathcal{F}_U$, $U \subseteq X$. Moreover, $\mathcal{F}_X$ is the trivial structure.

Lemma 4.13. *The following statements hold for any set X.*

(i) *If $A, B \subseteq X$ then $\mathcal{F}_A \in \mathbf{H}(\mathcal{F}_B)$ if and only if $B \subseteq A$.*
(ii) *Let $\{A_i \mid i \in I\} \subseteq X$ and let $A = \bigcap_{i \in I} A_i$. Then the structure $\mathcal{A} = \mathcal{F}_A \in \mathbf{K}'$ is isomorphic to a substructure in $\mathcal{B} = \prod_{i \in I} \mathcal{F}_{A_i}$.*

Let $\Sigma(X, C)$ consist of the following (in general infinite) implications of the form

$$\bigwedge_{u \in U} c_u = c \to c_v = c, \qquad U \subseteq X,\ v \in C(U).$$

Of course, if the set X is finite, then the signature $\sigma(X)$ is finite, while $\Sigma(X, C)$ becomes a finite set of quasi-identities. Let $\mathbf{K}(X, C) = \mathbf{K}' \cap \mathrm{Mod}\big(\Sigma(X, C)\big)$.

Proofs of all the results presented in this section are similar to ones of corresponding results about the class $\mathbf{K}_p(X, C)$ presented in Subsection 4.2.

Lemma 4.14. *For any closure space $\langle X, C\rangle$, the class $\mathbf{K}(X, C)$ consists of isomorphic copies of structures $\mathcal{F}_B$, where $B \in \mathbb{L}(X, C)$.*

Proposition 4.15. *For any complete lattice L, there is a signature σ consisting of $|L|+1$ many constant symbols and there is a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ such that $L \cong \mathrm{Lv}\big(\mathbf{K}\big)$ and $\mathrm{Sub}_c(L^\partial) \cong \mathrm{Lp}\big(\mathbf{K}\big)$.*

Proposition 4.16. *For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting of $|S|+1$ many constant symbols and there is a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ such that $\mathrm{Sub}(S) \cong \mathrm{Lp}^\omega(\mathbf{K})$.*

5. Relatively axiomatizable classes of structures

In [22, Theorem 8], D. E. Pal'chunov has proved that any at most countable complete lattice is isomorphic to a lattice of relatively axiomatizable classes. In [22, Problem 1], he asked whether the same result holds for an *arbitrary* complete lattice. M. Semenova and A. Zamojska-Dzienio answered the latter question in the positive in [25] for a signature consisting of unary relation symbols and a prevariety of trivial structures, see Theorem 5.2 below. We emphasize that this positive answer follows essentially by results of V. A. Gorbunov [9], see also [11] and Proposition 4.2.

Exposition here follows [25]. We also note that Theorem 5.2 can be inferred from results of Subsections 4.2-4.3 for a signature containing one unary relation symbol and constants as well as for a signature containing only constants.

Definition 5.1. [22, Definition 26] Let $\mathbf{K}$ be a class of structures of signature σ and let Δ be a set of first-order sentences of the same signature. A class $\mathbf{K}'$ is *axiomatizable in* $\mathbf{K}$ *relatively to* Δ, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set $\Sigma \subseteq \Delta$.

It follows from Definition 5.1 that a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is axiomatizable if and only if it is axiomatizable in $\mathbf{K}(\sigma)$ relatively to the set of all first-order sentences. Furthermore, for any set Δ of sentences and any class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, the set of all axiomatizable in $\mathbf{K}$ relatively to Δ classes forms a complete lattice. Following D. E. Pal'chunov [22], we denote this lattice by $\mathbb{A}(\mathbf{K}, \Delta)$. The following corollary shows that any complete lattice is a lattice of relatively axiomatizable classes.

Theorem 5.2. *For any complete lattice L, there are a signature σ, a prevariety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, and a set Δ such that $L \cong \mathbb{A}(\mathbf{K}, \Delta)$, where Δ is the set of all identities of signature σ.*

Now, we get from Corollary 4.5 and [11, Proposition 5.1.1]:

Corollary 5.3. *The class of complete dually algebraic lattices coincides with the class of lattices of the form $\mathbb{A}(\mathbf{K}, \Delta)$, where $\mathbf{K}$ is a quasivariety and Δ is a set of first-order sentences.*

Corollary 5.4. [22, Theorem 9], [25] *For any finite lattice L, there are a finite signature σ and a set Δ of first-order sentences of σ such that $L \cong \mathbb{A}(\mathbf{K}(\sigma), \Delta)$.*

6. A reduction theorem

In [10] (see also his monograph [11]), V. A. Gorbunov has proved so-called *reduction theorems* for lattices of quasivarieties and lattices of varieties. For a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, and for a positive $n < \omega$, let $\mathcal{F}_{\mathbf{K}}(n)$ denote the $\mathbf{K}$-free structure of rank n.

Theorem 6.1. [11, Corollaries 5.5.2, 5.5.12] *Let $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ be a prevariety. Then the following holds:*

$$\mathrm{Lq}(\mathbf{K}) \cong \varprojlim \mathrm{Lq}\big(\mathbf{H}(\mathcal{F}_{\mathbf{K}}(n)) \cap \mathbf{K}\big) \cong \varprojlim \mathrm{Sp}\big(\mathrm{Con}_{\mathbf{K}}\, \mathcal{F}_{\mathbf{K}}(n), \mathrm{E}\big);$$

$$\mathrm{Lv}(\mathbf{K}) \cong \varprojlim \mathrm{Lv}\big(\mathbf{H}(\mathcal{F}_{\mathbf{K}}(n)) \cap \mathbf{K}\big) \cong \varprojlim \mathrm{F}^*\big(\mathrm{Con}_{\mathbf{K}}\, \mathcal{F}_{\mathbf{K}}(n)\big).$$

In particular, the following statements are true.

Corollary 6.2. [11, Corollaries 5.5.4, 5.5.13] *Let σ contain finitely many relation symbols and let $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ be a locally finite prevariety. Then*

(i) $\mathrm{Lq}(\mathbf{K}) \cong \varprojlim L_n$ *for a set* $\{L_n \mid n < \omega\}$ *of finite lower bounded lattices;*
(ii) $\mathrm{Lv}(\mathbf{K}) \cong \varprojlim L_n$ *for a set* $\{L_n \mid n < \omega\}$ *of finite lattices.*

In particular, both $\mathrm{Lq}(\mathbf{K})$ *and* $\mathrm{Lv}(\mathbf{K})$ *are residually finite lattices.*

In [10], V. A. Gorbunov has also proved the following version of the reduction theorem for lattices of pseudo-quasivarieties.

Theorem 6.3. [11, Theorem 5.5.16] *Let σ contain only finitely many relation symbols and let $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ be a pseudo-quasivariety. Then there is a family $\{L_n \mid n < \omega\}$ of finite lower bounded lattices such that* $\mathrm{Lp}(\mathbf{K}) \cong \varprojlim L_n$.

In [25] M. Semenova and A. Zamojska-Dzienio proved a [finitary] prevariety analogue of Theorem 6.1. More precisely, the lattice of subprevarieties of a prevariety is isomorphic to an inverse limit of complete subsemilattice lattices of semilattices endowed with a distributive binary relation (see Theorem 6.4), while the lattice of finitary subprevarieties of a finitary prevariety is isomorphic to an inverse limit of subsemilattice lattices of semilattices endowed with a distributive binary relation (see Theorem 6.5). These results generalize Theorem 6.3.

To prove Theorems 6.4 and 6.5, one should assume the following class form of Axiom of Choice, see (CAC 1) in H. Rubin and J. E. Rubin [24, Section II.2]:

If $\mathbf{S}$ *is a class of non-empty sets,*
there is a function F *such that* $F(x) \in x$ *for each* $x \in \mathbf{S}$.

Theorem 6.4. [25] *For any prevariety* $\mathbf{K} \subseteq \mathbf{K}(\sigma)$*, the lattice* $\mathrm{Lp}(\mathbf{K})$ *is isomorphic to an inverse limit of lattices of the form* $\mathrm{Sub}_c(S, R)$*, where S is a complete meet semilattice with unit and R is a distributive relation on S.*

Sketch of proof. Let I be the class of all subsets of $\mathbf{K}$ ordered by inclusion, let $\mathcal{A}_i = \prod\{\mathcal{A} \mid \mathcal{A} \in i\}$, and let $\mathbf{K}_i = \mathbf{H}(\mathcal{A}_i) \cap \mathbf{K}$ for all $i \in I$. Moreover, as $\mathbf{K}_i \subseteq \mathbf{K}_j$, the map

$$\varphi_{ji}\colon\ \mathrm{Lp}(\mathbf{K}_j) \to \mathrm{Lp}(\mathbf{K}_i), \quad \varphi_{ji}\colon \mathbf{X} \mapsto \mathbf{X} \cap \mathbf{K}_i$$

is a complete lattice homomorphism for all $i \subseteq j$ in I. Besides that, $\varphi_{kj}\varphi_{ji} = \varphi_{ki}$ and φ_{ii} is just the identity map for all $i \subseteq j \subseteq k$ in I. Therefore, the triple $\Lambda = \langle I, \mathbf{K}_i, \varphi_{ji}\rangle$ is an inverse spectrum.

Now, the map $\varphi\colon\ \mathrm{Lp}(\mathbf{K}) \to \varprojlim \Lambda$ defined as follows:

$$\varphi\colon \mathbf{X} \mapsto \langle \mathbf{X} \cap \mathbf{K}_i \mid i \in I\rangle,$$

is a complete lattice isomorphism and one obtains $\mathrm{Lp}(\mathbf{K}) \cong \varprojlim \Lambda$.

Finally, for any $i \in I$, we have $\mathrm{Lp}(\mathbf{K}_i) = \mathrm{Lp}\big(\mathbf{H}(\mathcal{A}_i) \cap \mathbf{K}\big) \cong \mathrm{Sub}_c\big(\mathrm{Con}_{\mathbf{K}}\,\mathcal{A}_i, \mathrm{E}\big)$ according to Theorem 3.1, whence the statement of the theorem follows. $\square$

The next statement is an analogue of Theorem 6.4 for finitary prevarieties.

Theorem 6.5. [25] *For any finitary prevariety* $\mathbf{K} \subseteq \mathbf{K}(\sigma)$*, the lattice* $\mathrm{Lp}^{\omega}(\mathbf{K})$ *is isomorphic to an inverse limit of lattices of the form* $\mathrm{Sub}(S, R)$*, where S is a meet semilattice with unit and R is a distributive relation on S.*

Now Theorem 6.3 becomes an easy corollary of any of Theorems 6.4 and 6.5 according to the definition of a pseudo-quasivariety. We also note that to prove Theorem 6.5 for pseudo-quasivarieties, ordinary Axiom of Choice is sufficient.

It is not hard to check (see [11, Lemma 5.5.17 and Corollary 5.5.18]) that if $\mathbf{K}$ is a locally finite quasivariety, then the map

$$\varphi\colon \operatorname{Lq}(\mathbf{K}) \to \operatorname{Lp}(\mathbf{K}_{fin}); \quad \varphi\colon \mathbf{X} \to \mathbf{X}_{fin}$$

defines an isomorphism. Therefore, Theorem 6.4 implies Corollary 6.2(i).

For a pseudo-quasivariety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, let I be the set of all finite subsets of $\mathbf{K}$, let $\mathbf{K}_i = \mathbf{H}\bigl(\prod\{\mathcal{A} \mid \mathcal{A} \in i\}\bigr) \cap \mathbf{K}$ for all $i \in I$, and let

$$\mathbf{L}_q = \{\operatorname{Lq}\bigl(\mathbf{Q}(\mathbf{K}_i)\bigr) \mid i \in I\}.$$

The following corollary generalizes V. A. Gorbunov [11, Corollary 5.5.22].

Corollary 6.6. *Let σ contain finitely many relation symbols and let $\mathbf{K} \subseteq \mathbf{K}(\sigma)_{fin}$ be a pseudo-quasivariety. Then $\operatorname{Lp}(\mathbf{K}) \in \mathbf{SP}_u\mathbf{H}(\mathbf{L}_q) \cap \mathbf{SP}_u\bigl(\operatorname{Lq}\bigl(\mathbf{Q}(\mathbf{K})\bigr)\bigr)$. In particular, any universal sentence which holds in $\operatorname{Lq}\bigl(\mathbf{Q}(\mathbf{K})\bigr)$ also holds in $\operatorname{Lp}(\mathbf{K})$.*

The next theorem shows that a similar result for lattices of pseudo-varieties also holds. It was proved by P. Agliano and J. B. Nation [4] for pseudo-varieties of algebras, but their proof remains valid for structures with finitely many relation symbols.

Theorem 6.7. [4, Theorem 2.1] *Let σ contain finitely many relation symbols and let $\mathbf{K} \subseteq \mathbf{K}(\sigma)_{fin}$ be a pseudo-variety. Then the lattice $\operatorname{Lpv}(\mathbf{K})$ of pseudo-varieties containing in $\mathbf{K}$ belongs to the class*

$$\mathbf{HSP}_u\bigl(\operatorname{Lv}\bigl(\mathbf{V}(\mathcal{A})\bigr) \mid \mathcal{A} \in \mathbf{K}\bigr).$$

In particular, any positive universal sentence which holds in $\operatorname{Lv}\bigl(\mathbf{V}(\mathbf{K})\bigr)$ also holds in the lattice $\operatorname{Lpv}(\mathbf{K})$ of all pseudo-varieties contained in $\mathbf{K}$.

7. Non-computability properties of relative subclass lattices

The following problem which is due to G. McNulty [17]: *Is the set of all finite lattices of varieties computable?* This problem is also mentioned in W. A. Lampe [15].

In [21, Theorem 1], A. M. Nurakunov has proved the following statement.

Theorem 7.1. *Let a signature σ contain at least one non-constant operation. Then there is a quasivariety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ such that the set of all finite sublattices of the quasivariety lattice $\operatorname{Lq}(\mathbf{K})$ is not computable.*

The latter result means that there is no algorithm to decide whether a given finite lattice embeds into such a quasivariety lattice. Therefore, it looks hopeless to find a complete structural description of lattices isomorphic to [quasi]variety lattices, cf. the Birkhoff-Maltsev problem.

We also note that from the proof of Theorem 7.1, it is possible to get an estimation of algorithmic complexity for certain quasivariety lattices as well as to compute the number of non-isomorphic quasivariety lattices having a non-computable set of finite sublattices.

Corollary 7.2. *There is a locally finite quasivariety such that the set of all finite sublattices of its quasivariety lattice is not computable, while it is computably enumerable.*

Corollary 7.3. *There are continuum many locally finite quasivarieties such that the set of finite sublattices of their quasivariety lattices is not computable.*

While Theorem 7.1 and Corollaries 7.2-7.3 deal with purely functional signature, there are their complete analogues for purely relational signature. In particular, it is proved in [25] (based on ideas from [21]) that there are quasivarieties of one-element relation structures such that their [quasi]variety lattices or [finitary] prevariety lattices have a non-computable computably enumerable set of finite sublattices.

Theorem 7.4. [25] *The following statements hold.*

(i) *There is a countable relation signature τ and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\tau)$ such that the set of all finite sublattices of the relative variety lattice* $\mathrm{Lv}(\mathbf{K})$ *is computably enumerable but not not computable* [*not computably enumerable, respectively*].

(ii) *There is a countable relation signature σ and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that* $\mathrm{Lq}(\mathbf{K}) = \mathrm{Lp}(\mathbf{K}) = \mathrm{Lp}^{\omega}(\mathbf{K})$ *and the set of all finite sublattices of this lattice is computably enumerable but not computable* [*not computably enumerable, respectively*].

8. Open problems

As it has been already mentioned in Introduction, very little is known about lattices of first-order axiomatizable classes different from [quasi]varieties. Thus the following general problem arises:

Problem 1. Study lattices of [relatively] axiomatizable classes and lattices of [finitary] prevariety lattices.

Remark 4.7 suggests the following problem.

Problem 2. [25] Is there a nontrivial lattice property satisfied by all lattices of [finitary] prevarieties? Which lattices are isomorphic to lattices of [finitary] prevarieties?

Problem 2 is an analogue of the Birkhoff-Maltsev problem. It is well known (cf. R. Freese, J. Ježek, J. B. Nation [7, Theorem 2.84]) that finite bounded lattices generate the variety of all lattices. According to V. B. Repnitskiĭ [23], the lattice $\mathrm{Sub}_c(L)$ is finite lower bounded for any finite lattice L. Therefore, prevariety lattices of quasivarieties generate the variety of all lattices. Thus according to Proposition 4.2, there is no nontrivial lattice identity which would hold on all prevariety lattices.

Due to results presented in Section 7, one can also pose the following

Problem 3. For certain classes of structures, is the finite membership problem decidable?

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