Coevolutionary Algorithm for Building Robust Decision Trees under Minimax Regret

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Abstract

In recent years, there has been growing interest in developing robust machine learning (ML) models that can withstand adversarial attacks, including one of the most widely adopted, efficient, and interpretable ML algorithms—decision trees (DTs). This paper proposes a novel coevolutionary algorithm (CoEvoRDT) designed to create robust DTs capable of handling noisy high-dimensional data in adversarial contexts. Motivated by the limitations of traditional DT algorithms, we leverage adaptive coevolution to allow DTs to evolve and learn from interactions with perturbed input data. CoEvoRDT alternately evolves competing populations of DTs and perturbed features, enabling construction of DTs with desired properties. CoEvoRDT is easily adaptable to various target metrics, allowing the use of tailored robustness criteria such as minimax regret. Furthermore, CoEvoRDT has potential to improve the results of other state-of-the-art methods by incorporating their outcomes (DTs they produce) into the initial population and optimize them in the process of coevolution. Inspired by the game theory, CoEvoRDT utilizes mixed Nash equilibrium to enhance convergence. The method is tested on 20 popular datasets and shows superior performance compared to 4 state-of-the-art algorithms. It outperformed all competing methods on 13 datasets with adversarial accuracy metrics, and on all 20 considered datasets with minimax regret. Strong experimental results and flexibility in choosing the error measure make CoEvoRDT a promising approach for constructing robust DTs in real-world applications.

Introduction

Decision trees (DTs) is a popular, easily interpretable machine learning (ML) algorithm for classification and regression tasks. One of the primary challenges in DT construction is dealing with noisy and high-dimensional data. In particular, it has been shown that ML models (including DTs) are vulnerable to adversarial, perturbed samples that trick the model into misclassifying them (Kantchelian, Tygar, and Joseph 2016; Zhang, Zhang, and Hsieh 2020; Grosse et al. 2017). To address this challenge, researchers have proposed new defensive algorithms for creating *robust* classification models (see, e.g., Chakraborty et al. (2021)). A model is defined to be robust to some perturbation range of its input samples when it assigns the same class to all the samples within that perturbation range, so that small malicious alterations of input objects should not deceive a robust classifier.

The vast majority of defensive algorithms for DTs focus on adversarial accuracy (Kantchelian, Tygar, and Joseph 2016; Chen et al. 2019; Guo et al. 2022; Ranzato and Zanella 2021; Justin et al. 2021). We argue that there are better metrics in principle, and the focus on adversarial accuracy has been driven by computational tractability. Adversarial accuracy is highly sensitive to accuracy on the worst-case perturbation, and when the perturbation range can be large, this can lead to a flattening of intuitively good models and bad ones, as the worst-case perturbations can "defeat" all models.

There are other metrics that can better evaluate model robustness, like *max regret* (Savage 1951). Max regret is defined as the maximum difference between the result of the given model and the result of the optimal model for any input data perturbation within a given range. Minimizing max regret is referred to as the minimax regret decision criterion. Adversarial accuracy might provide an overly optimistic or pessimistic view of the model's robustness by focusing only on absolute accuracy value. In contrast, max regret is a more realistic approach since it counts the magnitude of the potential loss by considering the model trained on perturbed data. However, max regret cannot be directly optimized and used as a splitting criterion in the state-of-the-art algorithms.

In recent years, researchers successfully explored the potential of coevolutionary algorithms to various optimization problems (Mahdavi, Shiri, and Rahnamayan 2015) including DTs induction (Aitkenhead 2008). Coevolutionary algorithms consist in simultaneous evolution of multiple populations, each of them representing a different aspect of the problem. By fostering competition between populations, coevolutionary algorithms can guide the search towards the optimal solutions.

Considering the limitations of traditional DT algorithms and the promises of coevolutionary computation, we propose a novel coevolutionary algorithm specifically tailored for creating robust decision trees (RDTs) in adversarial contexts. Our approach leverages the power of adaptive coevolution, allowing to exploit the competitive interactions between populations of decision trees and adversarial perturbations to adapt and converge toward robust and accurate classifications for complex and noisy data. In this process, we can freely define robustness metrics to optimize (including max regret) which leads to the models better tailored to han-

dle perturbed high-dimensional data. Because of the inherent flexibility of evolutionary methods, we can additionally integrate other objective criteria, such as fairness (Aghaei, Azizi, and Vayanos 2019; Jo et al. 2022).

The main contribution of this paper is proposition of a novel coevolutionary algorithm (CoEvoRDT) capable of creating robust decision trees. CoEvoRDT has the following key properties:

- supremacy over state-of-the-art (SOTA) solution on 13 out of 20 datasets with adversarial accuracy metric and on all 20 datasets with minimax regret,
- predominance over existing evolutionary-based approaches for RDTs construction,
- to the best of our knowledge, it is the first algorithm able to directly optimize *minimax regret* for RDTs,
- it employs novel game theoretic approach for constructing the Hall of Fame with Mixed Nash Equilibrium,
- the algorithm is easily adaptable to various target metrics,
- by design, CoEvoRDT can be used for potential improvement of results of other SOTA methods by including their resulting DTs in the initial population and optimizing them through coevolution.

Problem definition

Let $X \subset \mathbb{R}^d$ be a d-dimensional instance space (inputs) and Y be the set of possible classes (outputs). A classical classification task is to find a function (model) $h: X \to Y$, $h(x_i) = y_i$, where y_i is true class of x_i . Classification performance of model h can be measured by accuracy:

$$\operatorname{acc}(h) = \frac{1}{|X|} \sum_{x_i \in X} I[h(x_i) = y_i],$$

where $I[h(x_i) = y_i]$ returns 1 if h predicts the true class of x_i , and 0, otherwise.

Let $\mathcal{N}_{\varepsilon}(x)=\{z:||z-x||_{\infty}\leq \varepsilon\}$ be a ball with center x and radius ε under the L_{∞} metric. The **adversarial accuracy** of a model h is accuracy on the perturbation in the perturbation set that produces the lowest accuracy. It is formally defined as

$$\operatorname{acc}_{\operatorname{adv}}(h, \epsilon) = \frac{1}{|X|} \sum_{x_i \in X} \min_{z_i \in \mathcal{N}_{\varepsilon}(x_i)} I[h(z_i) = y_i].$$

The **max regret** of a model h is the maximum regret among all possible perturbations $z \in \mathcal{N}_{\varepsilon}$. Regret is the difference between the best accuracy possible on a particular perturbation and the accuracy h achieves:

regret
$$(h, \{z_i\}) = \max_{h'} acc(h', \{z_i\}) - acc(h, \{z_i\}),$$

where $acc(h, \{z_i\})$ is the accuracy achieved by h when $\{x_i\}$ is replaced with $\{z_i\}$. Max regret be expressed as:

$$\operatorname{mr}(h) = \max_{z_i \in \mathcal{N}_{\varepsilon}(x_i)} \operatorname{regret}(h, \{z_i\})$$

The problem addressed in this paper is finding a DT trained on X that for a given ε optimizes (maximizes for adversarial accuracy or minimizes for max regret) a given robustness metric (one of the two above-mentioned).

Motivating example

Consider a financial institution that makes loan acceptance decisions. DTs are well-suited for such high-stakes scenario (Alaradi and Hilal 2020). The dataset of loan applicants in Figure 1 has two features: income I and credit score CS. The system should correctly make a binary credit decision D: accept (1) or reject (0).

For the data in T1, a simple one-node DT can achieve 100% accuracy. One possible decision rule to achieve this is $CS \geq 55$, which we call DT1.

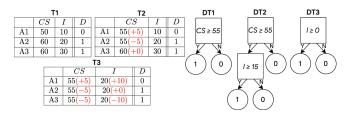


Figure 1: Motivational example – perturbed input data and 3 decision trees.

The features of training data may not be representative of test data due to bugs in the system, inaccurate input data, or distribution shift. Table T2 shows a potential perturbation of the data in T1. In T2, DT1 misclassifies A1 (returning 1 instead of 0). A more robust decision tree, DT2, accurately classifies all applicants in T1 and T2.

T3 is an example with a larger perturbation that affects both income and credit score. In this perturbation, the three applicants have the same features, meaning that no DT can classify them correctly. Thus, the adversarial accuracy against any perturbation set that includes T3 is at most $\frac{2}{3}$. However, achieving such accuracy is easy, any DT that always predicts 1 will do so, including the decision rule $I \geq 0$ (DT3). Consequently, for methods optimizing the adversarial accuracy metric, DT3 is one of the optimal solutions, but it is neither robust nor desired. Maximizing adversarial accuracy myopically focuses on the hardest perturbations in the perturbation set.

From a max regret perspective, DT2 outperforms DT3. Max regret considers not only the worst-case perturbation accuracy, but also the difference between the accuracy of the optimal DT and the robust DT for every perturbation. The regret of DT2 is 0 on all three datasets, resulting in a max regret of 0. DT3 achieves regret of $\frac{1}{3}$ on T1 and T2 and 0 on T3, resulting in a max regret of $\frac{1}{3}$. Thus, under the minimax regret criteria, DT2 would be selected over DT3. Adversarial accuracy loses its ability to distinguish between models as perturbations become large—intuitively good and bad can achieve the same scores.

Related work

There has been substantial recent work on the construction of robust decision trees. One line of work aims to improve robustness by choosing more appropriate splitting criteria. RIGDT-h (Chen et al. 2019) constructs robust DTs based on

the introduced notion of adversarial Gini impurity, a modification of classical Gini impurity (Breiman 2017) adapted to perturbed input data. This method was further improved in the GROOT algorithm (Vos and Verwer 2021), which mimics the greedy recursive splitting strategy that traditional DTs use and scores splits with the adversarial Gini impurity. The most recent approach, Fast Provably Robust Decision Trees (FPRDT) (Guo et al. 2022) is a greedy recursive approach to a direct minimization of the adversarial loss. It uses the 0/1 loss, rather than traditional surrogate losses such as square loss and softmax loss, as the splitting criterion during the construction of the DT. In experiments, we compare to Guo et al. (2022) and refer the reader to that paper for comparisons with prior methods. Max regret cannot be directly optimized by these methods, which leverage specific properties of adversarial accuracy to design splitting criteria.

Ranzato and Zanella (2021) introduced a genetic adversarial training algorithm (Meta-Silvae) to optimize DT stability, building on a history of using genetic algorithms for DTs Barros et al. (2011), and leveraging the geometric of adversarial accuracy. Our approach utilizes a coevolutionary method and, to the best of our knowledge, it is the first application of this technique to creating robust DTs. At the same time, the effectiveness of coevolutionary algorithms was experimentally proven in many other domains including multi-objective optimization (Meneghini, Guimaraes, and Gaspar-Cunha 2016; Tian et al. 2020), non-cooperative games (Razi, Shahri, and Kian 2007; Żychowski and Mańdziuk 2023), preventing adversarial attacks (O'Reilly and Hemberg 2018), or Generative Adversarial Networks training (Costa, Lourenço, and Machado 2019; Toutouh et al. 2023).

An alternative, exact, approach to robust DTs proposed by Justin et al. (2021) uses a mixed-integer optimization formulation. However, at present, its present applicability is limited to small datasets (approximately less than 3200 samples and/or up to 36 features).

Although, to the best of our knowledge, max regret has not been specifically studied in DTs, it was applied to other domains, e.g., neural network training (Alaiz-Rodriguez, Guerrero-Curieses, and Cid-Sueiro 2007), reinforcement learning (Azar, Osband, and Munos 2017; Xu et al. 2021), robust planning in uncertain Markov decision processes (Rigter, Lacerda, and Hawes 2021), Security Games (Nguyen et al. 2014), and computing randomized Nash equilibrium (Gilbert and Spanjaard 2017).

CoEvoRDT algorithm

A general overview of the Coevolutionary method for Robust Decision Trees (CoEvoRDT) is presented in Algorithm 1. CoEvoRDT maintains two populations: one contains encoded DTs, and the other contains input data perturbations. Both populations are initialized with random elements and then developed alternately. First, the DT population is modified by evolutionary operators (crossover, mutation, and selection) through l_c generations. Then, the perturbation population is evolved through the same number of l_c generations. The above loop is repeated until the stop condition is satisfied.

Algorithm 1: CoEvoRDT pseudocode.

```
1: P_T \leftarrow InitializeDecisionTreesPopulation()
 2: P_P \leftarrow InitializePerturbationsPopulation()
 3: HoF_T = HoF_P = \emptyset // HoF - Hall of Fame
 4: N_{\text{top}} = 20
 6: while stop condition not satisfied do
 7:
          for 1..l_c do
 8:
               P_T \leftarrow P_T \cup \text{Crossover}(P_T)
               P_T \leftarrow P_T \cup \text{Mutate}(P_T)
 9:
10:
              P_T \leftarrow \text{Evaluate}(P_T, P_P, \text{HoF}_P)
              P_T^* \leftarrow \text{GetElite}(P_T)
11:
12:
              while |P_T^*| < N_T do
                   P_T^* \leftarrow P_T^* \cup \text{BinaryTournament}(P_T)
13:
              end while
14:
15:
              P_T \leftarrow P_T^*
16:
              \mathcal{T}, \mathcal{P} \leftarrow \text{MixedNashEquilibrium}(P_T, P_P)
17:
              \text{HoF}_T \leftarrow \text{HoF}_T \cup \mathcal{T}
18:
              \text{HoF}_P \leftarrow \text{HoF}_P \cup \mathcal{P}
19:
           end for
20:
21:
          for 1..l_c do
22:
              P_P \leftarrow P_P \cup \text{Crossover}(P_P)
              P_P \leftarrow P_P \cup \text{Mutate}(P_P)
23:
24:
              P_P \leftarrow \text{Evaluate}(P_P, P_T, \text{HoF}_T, N_{\text{top}})
25:
              P_P^* \leftarrow \text{GetElite}(P_P)
26:
              while |P_P^*| < N_P do
27:
                   P_P^* \leftarrow P_P^* \cup \text{BinaryTournament}(P_P)
28:
              end while
              P_P \leftarrow P_P^*
29:
              \mathcal{T}, \mathcal{P} \leftarrow \text{MixedNashEquilibrium}(P_T, P_P)
30:
31:
              \text{HoF}_T \leftarrow \text{HoF}_T \cup \mathcal{T}
32:
              \text{HoF}_P \leftarrow \text{HoF}_P \cup \mathcal{P}
33:
          end for
34: end while
35:
36: return \arg \max_{t \in P_{\mathcal{T}}} \xi(t)
```

Decision tree population

The DT population contains N_T individuals. Each individual represents one candidate solution (DT) which is encoded as a list of nodes. Each node is represented by a 7-tuple: $node = \{t, c, P, L, R, o, v, a\}$, where t is a node number (t=0) is the root node), c is a class label of a terminal node (meaningful only for terminal nodes), P is a pointer to the parent node, L and R are pointers to the left and right children, respectively (null in a terminal node), o indicates which operator is to be used (<,>,=) and v is a real number that indicates the value to be tested on attribute a.

The **initial population** consists of trees generated by randomly choosing attributes and split values, and halting the growth of each DT when the tree reaches a depth randomly selected from an interval [2, 10].

Each individual from the population is selected for **crossover** with probability p_c . Selected individuals are paired randomly, and the crossover operator selects random nodes in two individuals and exchanges the entire subtrees corresponding to each selected node, generating two offspring individuals which are added to the current population.

The **mutation** operator introduces random changes to the

individuals. Each individual is mutated with probability p_m . The mutation operator applies randomly one of the three following actions: (i) replacing a subtree with a randomly generated one, (ii) changing the information in a randomly selected node (setting a new random splitting value v or operator o), (iii) prune a randomly selected subtree. For each mutated individual the mutation is applied 10 times and the highest-fitness individual (among these 10) is added to the current population.

The **evaluation** procedure is performed against the perturbation population (described next). For each individual (a candidate DT) the metric being optimized is computed against all perturbations from the adversarial population. Since the number of perturbations is relatively small any arbitrary chosen metric can be effectively calculated and assigned as an individual's fitness value.

Perturbation population

The perturbation population consists of N_P individuals $P \subset \mathbb{R}^d$. Each of them represents a perturbed input set X (one perturbation per instance), i.e., $\forall_{x \in X} \exists !_{p_x \in P} : p_x \in \mathcal{N}_{\varepsilon}(x)$.

The **initial population** contains random perturbations generated by drawing uniformly each perturbation element from the set of possible ones (according to ε criteria).

The **crossover** procedure selects a random subset of individuals (each individual is taken with probability p_c) and pairs them randomly. Then, for each pair, perturbed input instances from both individuals are mixed randomly, i.e., given two crossed parents $c^0=(x_1^0,\ldots,x_n^0)$ and $c^1=(x_1^1,\ldots,x_n^1)$ the offspring is $\overline{c}^0=(x_1^{i_1},\ldots,x_n^{i_n})$ and $\overline{c}^1=(x_1^{1-i_1},\ldots,x_n^{1-i_n})$, where $i_1,\ldots,i_n\in\{0,1\}$.

Mutation is applied with probability p_m independently to each individual. If a chromosome is selected for mutation, for each input instance and each attribute with probability 0.5 its encoded value is randomly perturbed, i.e. a new random feasible (according to ε constraint) value is assigned.

Evaluation of the perturbation individuals is not an obvious task. On the one hand, assigning the average accuracy versus all DTs in the DT population as a fitness value may be a weak approach. Observe that a given perturbation may be powerful only against a specific though relevant subset of the DTs, and as such should be preserved, but averaging across all DTs will decrease its fitness, posing a risk of omitting it in the selection process. On the other hand, if the perturbation fitness value was computed only against the best DT from the DT population, it would lead to an oscillation of the perturbation population. All perturbations would tend to be efficient for a particular DT, becoming vulnerable to other DTs and losing diversity. Thus, we use $N_{\mathrm{top}}=20$ highest-fitness DTs (merged with all DTs from Hall of Fame) to evaluate each perturbation and perform a targeted optimization with $N_{\rm top} = 1$ only if we would otherwise terminate (see Stop condition). Experimental justification of the above choice is presented in the supplementary material (SM).

Hall of Fame

Hall of Fame (HoF) is a mechanism used to retain and store the best-performing individuals (solutions) that have been encountered during the evolutionary process. By preserving them, the HoF prevents the loss of valuable information and ensures that the best-performing solutions are not discarded during the evolution. The most common approach is to add one of the highest-fitness individuals from each generation (Michalewicz 1996). We find this approach to suboptimal with respect to diversity. Although the HoF stores the best solutions, it can also be used to maintain a diverse set of high-performing individuals. Diversity is essential in evolutionary algorithms to avoid premature convergence, when the algorithm gets stuck in a local optimum and fails to explore better solutions. The HoF can promote diversity by storing solutions that represent different regions of the solution space.

In our coevolutionary approach, HoF is used to assess solutions more accurately. Namely, instead of calculating the fitness function only against individuals from the adversarial population, it is calculated against a merged set of HoF and population individuals.

Instead of adding the highest-fitness individual to the HoF, in CoEvoRDT, we use a game-theoretic approach. Decision trees and perturbation populations can be treated as sets of strategies of two players in a non-cooperative zero-sum game. Then, it is possible to calculate mixed Nash equilibrium. The result is the pair of mixed strategies, i.e., a subset of DTs from the population with assigned probabilities $\mathcal{T} = \{(T_1, p_{T_1}), \dots, (T_n, p_{T_n})\}$ and a similar subset of perturbations with probabilities \mathcal{P} $\{(P_1, p_{P_1}), \dots, (P_m, p_{P_m})\}$. Formally, a mixed Nash equilibrium is a pair $(\mathcal{T}, \mathcal{P})$ such as $\forall_{\mathcal{T}' \neq \mathcal{T}} \xi_T(\mathcal{T}', \mathcal{P}) \leq \xi_T(\mathcal{T}, \mathcal{P})$ and $\forall_{\mathcal{P}' \neq \mathcal{P}} \xi_P(\mathcal{T}, \mathcal{P}') \leq \xi_P(\mathcal{T}, \mathcal{P})$ where $\xi_{T|P}(\mathcal{T},\mathcal{P})$ denotes some objective robustness metrics calculated for a "mixed" decision tree \mathcal{T} and "mixed" perturbation \mathcal{P} (either adversarial accuracy or max regret, in our experiments). Note that this is zero-sum because, in robust optimization, the adversary aims to minimize the DT payoff (i.e., objective function): $\xi_T(\mathcal{T}, \mathcal{P}) = -\xi_P(\mathcal{T}, \mathcal{P})$. We add mixed strategies from Nash equilibria to both HoFs, and they are used in the evaluation process as described above. To evaluate a metric against a mixed object (tree or perturbation), we calculate the expected metric value—first computing the metric for each pair of pure strategies and then taking a weighted average according to the Nash equilibrium probabilities. We limit HoF size by the lowest-fitness element when a fixed maximum size is exceeded.

A similar approach was previously proposed in (Ficici and Pollack 2003) but instead of storing in HoF pure strategies from mixed Nash equilibrium we add mixed strategies. The intuition is that a mixed tree is more robust to diverse perturbations, which has a positive impact on the evolution of perturbations. Similarly, mixed perturbations force the DT population to create more robust DTs that are resistant to a wide spectrum of perturbed data. We demonstrate this in the experiments section.

Selection

The selection process decides which individuals from the current population will be promoted to the next generation. In the beginning, e individuals with the highest fitness value

are unconditionally transferred to the next generation. They are called *elite* and preserve the highest-fitness solutions. Then, a *binary tournament* is repeatedly executed until the next generation population is filled with N individuals. In each tournament, two individuals are sampled (with replacement) from the current population (including those affected by crossover and/or mutation). The higher-fitness chromosome (the winner) is promoted to the next generation with probability p_s (so-called selection pressure parameter). Otherwise, the lower-fitness one is promoted.

Stop condition

The algorithm ends when at least one of the following conditions is satisfied: (a) CoEvoRDT attains the maximum number of generations (l_a) , (b) no improvement of the best-found solution (DT) is observed in consecutive l_c generations. If condition (b) is satisfied, an additional local perturbation improvement subroutine is performed. This procedure is part of the stopping condition and aims to find a better perturbation for the best-fitness decision trees (DTs). Specifically, for each DT with the highest fitness value, the perturbation population evolves using the same process as outlined in lines 21–33 of the Algorithm 1, but with $N_{\rm top}=1$ (see line 24). This means that the evaluation for each perturbation is conducted against the DT with current highest fitness. If, for all of those DTs, this routine discovers a perturbation that decreases the fitness of the DT, the counter l_c is reset to zero, and the algorithm execution continues (with $N_{\rm top}=20$ and the population's state before local perturbation improvement subroutine execution supplemented with the newly found better perturbation).

To verify conditions (a) and (b) only generations of the DT population are considered. The highest-fitness DT is returned as a CoEvoRDT result.

Convergence

The alternating optimization can be understood as improving candidate DTs while progressively tightening a bound on the robust objective, as any set of perturbation provides a upper (resp., lower bound) on adversarial accuracy (resp., max regret). When $N_{\rm top}=1$ (e.g., at convergence), the bound of the objective has been tightened as much as possible for a candidate DT.

Theorem 1. If both the decision tree and the perturbation population contain an individual that maximizes their fitness against the opposing population, the decision tree in the current population with the highest fitness optimizes the robust objective.

Proof. Suppose the stop condition is met and fitness is maximized by the decision tree and perturbation populations. We claim that the highest-fitness decision tree h maximizes the robust objective. Suppose that this is not the case. Then, either (i) there exists a perturbation z that would lower h's fitness and is missing from the adversarial population or (ii) there is a decision tree h' that would have higher fitness than h but is missing. (ii) cannot occur because we assume that h maximizes fitness. (i) cannot happen because $N_{\rm top}=1$ when the stop condition is met.

Results and discussion

Experimental setup. The proposed method was tested on 20 widely used classification benchmark problems of various characteristics – the number of instances, features, and perturbation coefficient. All selected datasets were used in previous studies mentioned in the Related work section, and they are publicly available at https://www.openml.org. Table 1 summarizes their basic parameters.

dataset	ε	Instances	Features	Classes
ionos	0.2	351	34	2
breast	0.3	683	9	2
diabetes	0.05	768	8	2
bank	0.1	1372	4	2
Japan:3v4	0.1	3087	14	2
spam	0.05	4601	57	2
ĠesDvP	0.01	4838	32	2
har1v2	0.1	3266	561	2
wine	0.1	6497	11	2
collision-det	0.1	33000	6	2 2 2 2 2 2 2 2 2 2 2 2 2
mnist:1v5	0.3	13866	784	2
mnist:2v6	0.3	13866	784	2
mnist	0.3	70000	784	10
f-mnist:2v5	0.2	14000	784	2
f-mnist:3v4	0.2	14000	784	2
f-mnist:7v9	0.2	14000	784	2
f-mnist	0.2	70000	784	10
cifar10:0v5	0.1	12000	3072	2
cifar10:0v6	0.1	12000	3072	2
cifar10:4v8	0.1	12000	3072	2

Table 1: Basic parameters of the benchmark datasets.

The CoEvoRDT parameter values used in the experiments and their selection process is described in detail in the SM. Since there is no straightforward method to calculate the exact values of adversarial accuracy and minimax regret (due to the presence of infinitely many possible perturbations), the results presented below are computed based on a sample of 10⁵ random perturbations (the same set for each compared method). The reasoning behind choosing this particular sample size is explained in the SM.

We adopted the Lemke-Howson algorithm (Lemke and Howson 1964) for calculating Mixed Nash Equilibrium from Nashpy Python library (Knight and Campbell 2018). The CART (Breiman 2017) method was used for computing the reference tree for minimax regret (i.e., highest accuracy trees for a particular perturbation). Statistical significance was checked according to the paired t-test with p-value ≤ 0.05 . All tests were run on Intel Xeon Silver 4116 @ 2.10GHz. CoEvoRDT source code is made publicly available at https://github.com/zychowskia/CoEvoRDT.

Robustness. CoEvoRDT was trained separately for max regret and adversarial accuracy, and compared with 4 SOTA methods (discussed in the related work section). The results are shown in Tables 2 and 3, respectively. They were also compared with the CART algorithm (Breiman 2017), a popular method for creating DTs for non-perturbed training data (not designed for the RDT scenario). In both tables, the last column (CoEvoRDT+FPRDT) presents the results of adding the FPRDT output DT to the CoEvoRDT initial population, and running CoEvoRDT afterwards. On the max regret metric, CoEvoRDT clearly outperforms all other competitors on all datasets. Adding the FPRDT tree only narrowly improves its outcome. The results support our claim that SOTA methods, which cannot directly minimize max regret, perform

dataset	CART	Meta Silvae	RIGDT-h	GROOT	FPRDT	CoEvoRDT	CoEvoRDT+FPRDT
ionos	.094±.000	.075±.007	.071±.006	.061±.005	.061±.006	.052±.004	.052±.005
breast	.103±.000	$.056 \pm .006$	$.069 \pm .006$	$.059 \pm .005$	$.057 \pm .005$	$.049 \pm .004$	$.049 \pm .005$
diabetes	$.202 \pm .000$.126±.008	.132±.009	.124±.009	.117±.007	$.096 \pm .006$	$.094 \pm .007$
bank	.186±.000	.102±.007	.108±.008	$.090 \pm .006$	$.089 \pm .007$	$.076 \pm .006$	$.076 \pm .006$
Japan3v4	.107±.000	$.090 \pm .006$.083±.006	$.067 \pm .006$	$.066 \pm .004$	$.062 \pm .006$	$.061 \pm .006$
spam	$.097 \pm .000$	$.079 \pm .006$	$.083 \pm .006$	$.074 \pm .006$	$.074 \pm .006$	$.070 \pm .005$	$.069 \pm .005$
GesDvP	.152±.000	.129±.008	.133±.010	$.129 \pm .008$.131±.009	$.114 \pm .007$.114±.007
har1v2	.105±.000	.074±.006	.084±.007	.068±.006	.068±.006	$.064 \pm .005$	$.064 \pm .005$
wine	.140±.000	.125±.008	.127±.009	.111±.009	$.109 \pm .008$	$.090 \pm .006$	$.090 \pm .007$
collision-det	.142±.000	$.099 \pm .007$	$.093 \pm .007$	$.088 \pm .006$	$.091 \pm .007$	$.061 \pm .006$	$.059 \pm .006$
mnist:1v5	.249±.000	$.078 \pm .007$.076±.006	$.071 \pm .006$	$.067 \pm .005$	$.055 {\pm} .006$.055±.005
mnist:2v6	$.268 \pm .000$	$.083 \pm .007$	$.087 \pm .006$	$.072 \pm .005$	$.069 \pm .005$	$.055 \pm .004$	$.054 \pm .004$
mnist	.395±.000	.143±.009	.139±.009	.125±.007	.124±.009	$.113 \pm .008$.112±.008
f-mnist2v5	$.273 \pm .000$.254±.015	.249±.015	$.223 \pm .013$.238±.014	$.196 \pm .011$.196±.011
f-mnist3v4	$.290 \pm .000$.259±.014	.254±.015	.246±.014	.232±.013	$.202 \pm .011$.199±.011
f-mnist7v9	.283±.000	.255±.014	.251±.015	.237±.014	.240±.014	.208±.013	.207±.012
f-mnist	.427±.000	.345±.020	.337±.018	.292±.017	.286±.016	.238±.014	.237±.015
cifar10:0v5	.419±.000	.351±.019	.379±.021	.347±.019	.314±.018	.241±.015	.236±.013
cifar10:0v6	.403±.000	.362±.021	.368±.020	.342±.018	.341±.019	$.289 \pm .016$.289±.016
cifar10:4v8	.408±.000	.357±.019	.360±.021	.339±.018	.331±.019	.283±.016	.281±.017

Table 2: **Max regrets** (mean \pm std error). CoEvoRDT+FPRDT obtained the best results for all datasets. The best results (except CoEvoRDT+FPRDT) are **bolded**. Gray background indicates that a given method is statistically significantly better than all other methods (except CoEvoRDT+FPRDT).

dataset	CART	Meta Silvae	RIGDT-h	GROOT	FPRDT	CoEvoRDT	CoEvoRDT+FPRDT
ionos	.310±.000	.695±.039	.701±.045	.783±.047	.795±.047	.791±.044	.795±.049
breast	$.250 \pm .000$.797±.047	$.838 \pm .052$.874±.047	.876±.055	.885±.054	.889±.056
diabetes	.542±.000	.554±.035	$.569 \pm .033$	$.623 \pm .043$	$.648 \pm .039$.617±.038	.648±.037
bank	.633±.000	.510±.031	$.468 \pm .033$.541±.036	$.658 \pm .040$.657±.043	.663±.037
Japan3v4	$.576 \pm .000$.566±.035	$.564 \pm .037$.584±.035	$.667 \pm .039$.665±.037	.668±.037
spam	$.302 \pm .000$	$.637 \pm .036$	$.467 \pm .028$	$.723 \pm .045$	$.746 \pm .049$	$.751 \pm .049$.753±.045
GesDvP	.478±.000	.637±.039	$.548 \pm .033$.716±.045	$.735 \pm .040$	$.740 \pm .046$.741±.044
har1v2	.232±.000	.706±.045	$.707 \pm .047$.806±.048	$.804 \pm .049$	$.818 \pm .054$.820±.052
wine	.620±.000	.637±.039	$.474 \pm .027$.637±.036	.674±.037	$.688 \pm .046$.692±.047
collision-det	$.743 \pm .000$.772±.047	$.764 \pm .044$.784±.052	$.792 \pm .051$	$.798 \pm .053$.803±.049
mnist:1v5	$.921 \pm .000$.952±.056	$.957 \pm .054$.954±.056	$.966 \pm .058$.964±.059	.969±.061
mnist:2v6	$.862 \pm .000$.906±.054	$.919 \pm .050$.917±.052	$.922 \pm .049$.917±.053	.922±.051
mnist	.673±.000	.702±.041	$.704 \pm .042$	$.743 \pm .048$	$.742 \pm .049$	$.745 \pm .043$.754±.046
f-mnist2v5	$.675 \pm .000$.951±.053	$.945 \pm .060$.971±.057	.978±.055	$.982 {\pm} .055$.982±.059
f-mnist3v4	$.632 \pm .000$.808±.049	$.793 \pm .044$.819±.048	$.865 \pm .050$	$.869 \pm .056$	870±.054
f-mnist7v9	.642±.000	.824±.045	$.81 \pm .052$.829±.052	$.876 \pm .050$.868±.054	.880±.047
f-mnist	.464±.000	.492±.033	$.525 \pm .033$.536±.035	.531±.033	$.544 \pm .036$.546±.040
cifar10:0v5	.296±.000	.502±.033	$.347 \pm .026$.485±.036	.678±.046	$.685 \pm .039$.693±.039
cifar10:0v6	.587±.000	.540±.038	.477±.029	.556±.037	.688±.040	$.692 \pm .046$.697±.043
cifar10:4v8	.256±.000	.514±.032	.488±.033	.473±.032	.661±.042	$.663 \pm .045$.664±.037

Table 3: **Adversarial accuracies** (mean \pm std error). CoEvoRDT+FPRDT obtained the best results for all datasets. Box denotes that CoEvoRDT+FPRDT is statistically significantly better than all other methods. The best results (except CoEvoRDT+FPRDT) are **bolded**. Gray background indicates that a given method is statistically significantly better than all other methods (except CoEvoRDT+FPRDT).

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT		CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT		N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.304	.238	.237	.237	.531	.544	.546	.546	19	79	97	97
2	.302	.237	.236	.236	.535	.546	.548	.548	40	161	114	185
3	.301	.237	.236	.236	.539	.548	.549	.550	60	240	134	272
4	.300	.236	.236	.235	.545	.550	.552	.553	80	321	165	362
5	.299	.236	.235	.235	.548	.552	.554	.557	99	406	183	496
10	.293	.234	.233	.233	.552	.557	.559	.562	195	774	264	939
20	.284	.230	.230	.229	.558	.564	.566	.568	391	1553	454	1869
50	.282	.229	.229	.227	.563	.568	.568	.569	956	3863	999	4564
100	.282	.228	.229	.227	.566	.568	.568	.569	1921	7981	1990	9026

Table 4: Best results of repeated N algorithms' runs for **fashion-mnist** dataset. In CoEvoRDT + N FPRDT output DTs from N FPRDT independent runs were incorporated into CoEvoRDT initial population.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	тор к	Desi	tree	mixed tree	trees	Top K	Dest	tree	mixed tree	trees	тор к	Dest
0	.261	.261	.261	.261	.261	.533	.533	.533	.533	.533	47	47	47	47	47
10	.242	.248	.247	.251	.259	.535	.535	.535	.534	.533	50	50	50	50	50
20	.240	.246	.245	.249	.256	.536	.536	.536	.536	.534	55	54	55	56	51
50	.241	.244	.245	.249	.254	.536	.536	.536	.536	.534	61	58	59	62	54
100	.239	.243	.243	.247	.253	.538	.538	.537	.537	.535	68	63	66	65	56
200	.238	.242	.242	.244	.250	.543	.539	.540	.539	.535	77	70	76	77	59
500	.237	.241	.241	.243	.248	.545	.540	.540	.540	.536	86	79	91	90	60
∞	.237	.239	.240	.240	.248	.545	.540	.541	.540	.536	86	77	85	85	61

Table 5: Results with respect of HoF size for **fashion-mnist** dataset. ∞ means that there was no limit on HoF size.

significantly worse than CoEvoRDT in terms of this metric. From an adversarial accuracy perspective, for 13 out of 20 datasets, CoEvoRDT yielded the best mean results (5 of them statistically significant). For the remaining 7 datasets FPRDT method was superior (with statistical significance in 3 cases). For this metrics, adding FPRDT tree to the initial CoEvoRDT population (CoEvoRDT+FPRDT) led to clear advantage versus baseline CoEvoRDT and FPRDT alone.

For a more detailed analysis, fashion-mnist dataset is selected as one of the largest. Detailed results for all other datasets are presented in SM.

Runtime comparison. In general, CoEvoRDT runtime varies from a few seconds to a couple of minutes for the largest datasets. The average computation time of a single run of the strongest competitor, FPRDT is 2 to 8 times lower than CoEvoRDT. Thus, the natural question which can arise is what if we run FPRDT multiple times to have an equal computation budget and choose the best result. This approach is addressed in Table 4, which presents computation time and the best results in terms of minimax regret and adversarial accuracy for multiple runs of FPRDT, CoEvoRDT, and CoEvoRDT initialized with multiple FPRDT outcomes. More runs can notably improve results for all methods. For max regret, even within 100 repeats, FPRDT was not able to find a solution close to the single CoEvoRDT outcome. For adversarial accuracy multiple FPRDT runs outperformed CoEvoRDT, but for greater numbers of repeats (above 20) both methods seem to converge to similar results. Given some constrained computation budget the best option is to run CoEvoRDT + N FPRDT.

HoF size and construction. Table 5 presents the results for various HoF sizes. For each size, 5 variants of constructing HoF are considered: adding one mixed tree from mixed Nash equilibrium after each generation (which is the baseline used in CoEvoRDT), adding all single trees from mixed Nash equilibrium (i.e., all the pure strategies with positive probability), adding only one highest-fitness tree from the population, adding top K highest-fitness individuals from the current population (where K is the number of trees from Nash mixed equilibrium) and adding one mixed tree composed of top K highest-fitness trees with equal probabilities. Firstly, it is clear that even small HoF significantly improves results for all variants. Moreover, adding a Nash mixed tree seems to be the best option, while adding only one highestfitness tree is the worst approach. The advantage of Nash mixed tree and top K as a mixed tree with equal probabilities over Nash single trees and top K trees shows the benefit of using mixed trees in the HoF. It may stem from the fact that mixed tree is *more robust* to various perturbations, and consequently the perturbation population is forced to find better perturbations to *outplay* those mixed trees. As a result, the DT population is forced to create even more robust trees. At the same time, the straightforward approach of creating a mixed tree of highest-fitness individuals is less powerful than a mixed tree from mixed Nash equilibrium.

The **generation limit** of CoEvoRDT was set to 1000, but in practice, it was rarely reached and the other stop condition (no improvement of best-found solution) was fulfilled first. The average number of generations across all datasets was 385. The lowest average number was observed for the diabetes (152), and the highest for cifar10:0v5 (865). The depth of DTs generated by CoEvoRDT varies from 6 to 23 which is not much different than DTs created by other methods.

CoEvoRDT **memory consumption** is low and does not exceed 150 MB for the largest datasets.

Conclusions

In this paper, we present CoEvoRDT, a novel coevolutionary algorithm designed to construct robust decision trees capable of handling perturbed high-dimensional data. Our motivation stems from the vulnerability of traditional DT algorithms to adversarial perturbations and the limitations of existing defensive algorithms in optimizing specific metrics like max regret. The flexibility of CoEvoRDT in accommodating various target metrics makes it adaptable to a wide range of applications and domains, including when robustness is mixed with other objectives such as fairness. We propose a novel game-theoretic approach to constructing the Hall of Fame with Mixed Nash Equilibrium, which significantly contributes to the DTs robustness and convergence speed. CoEvoRDT can additionally integrate results from another strong and fast method into the initial population, if one is available, to further improve performance.

CoEvoRDT was comprehensively tested on 20 popular benchmark datasets and compared with 4 SOTA algorithms, presenting on par performance to the best competitive method in adversarial accuracy metrics, and outperforming all competitors in terms of minimax regret.

Our future work focuses on investigating the potential of implementing CoEvoRDT as a multi-population algorithm, such as the island model (Skolicki 2005), to speed up convergence and potentially further boost its performance.

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Coevolutionary Algorithm for Building Robust Decision Trees under Minimax Regret

Supplementary material

Computation times

Table 1 presents a computation times comparison for Co-EvoRDT and state-of-the-art methods. Clearly, the fastest (but with the weakest results) is CART which does not consider robustness. Evolutionary approaches (Meta Silvae and CoEvoRDT) are a few times slower than other methods which bases on splitting techniques more appropriate for creating robust decision trees. Datasets with the longest computation time are from *cifar* family because of the greatest number of features (3072) which makes for lots of potential splitting options as well as greatly increases possible perturbations space.

CoEvoRDT parameterization

CoEvoRDT parameterization process was performed on cod-rna dataset (Uzilov, Keegan, and Mathews 2006) with 9 features, 2 classes, and 48565 instances. This dataset was not used in CoEvoRDT experimental evaluation described in the main paper. The algorithm was executed 10000 times with parameter values set randomly, i.e. for each run each parameter was drawn uniformly from some predefined set of values:

- decision trees population size $N_T : \{10, 20, 50, 100, 200, 500, 1000\}$
- perturbations population size N_P : {100, 200, 500, 1000, 2000, 5000, **10000**}
- number of consecutive generations for each population $l_c: \{1, 2, 5, 10, \mathbf{20}, 50, 100\}$
- the number of the best individuals from the decision trees population involved in the perturbations evaluation $N_{\text{top}}: \{1, 2, 5, 10, \textbf{20}, 50, 100, 200\}$
- crossover probability $p_c: \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, \textbf{0.8}, 0.9, 1.0\}$
- mutation probability $p_m:\{0.0,0.1,0.2,0.3,0.4,\textbf{0.5},0.6,0.7,0.8,0.9,1.0\}$
- selection pressure $p_s: \{0.5, 0.6, 0.7, 0.8, \mathbf{0.9}, 1.0\}$
- HoF size

 $N_{\text{HoF}}: \{0, 10, 20, 50, 100, 200, \mathbf{500}\}$

• generations without improvement limit $l_c: \{5, 10, 20, \mathbf{50}, 100, 200\}$

• generations limit $l_q: \{100, 200, 500, 1000, 2000, 5000\}$

Best values (with the lowest average minimax regret across all runs) are bolded.

The following CoEvoRDT parameter values were set in the experiments and comparison with state-of-the-art methods: decision tree population size $N_T=200$, perturbation population size $N_P=500$, number of consecutive generations for each population $l_c=20$, number of best individuals from the DT population involved in the perturbations evaluation $N_{\rm top}=20$, crossover probability $p_c=0.8$, mutation probability $p_m=0.5$, selection pressure $p_s=0.9$, elite size e=2, HoF size $N_{\rm HoF}=200$, generations without improvement limit $l_c=50$, generations limit $l_q=1000$,

Below we present a more detailed analysis for 4 parameters which appeared to be the most interesting.

Figure 2 shows the results (average minimax regret and computation time) according to the number of individuals in the decision trees population. Clearly, the bigger the population size the better the results since more potential solutions were checked. However, the improvement for large populations (500 and 1000 individuals) is insignificant compared to additional computation time which needs to be dedicated. Thus, the best trade-off between results quality and computation time seems to be $N_T=200$ and this value was adopted as a recommended value.

A similar relationship can be observed for perturbations population size in Figure 3. More perturbations mean more accurate assessment for decision trees from the adversarial population but it also costs more computational power. As in the previous case $N_P=500$ was chosen as a good compromise between minimax regret and computation time.

The results of tuning l_c parameter are presented in Figure 4. Small values ($l_c \leq 5$ - frequent switching between populations), as well as big ones ($l_c \geq 50$) result in performance deterioration. Infrequent switching makes one population dominant and the other one stagnates over a long time with no chances to respond to the evolved individuals from the other population. On the other hand, in frequent switching the population may not have enough generations to find adequate response to adversarial individuals. At the same time, for all tested values the computation time is similar. Hence, $l_c = 20$ was adopted as a recommended value.

The next tuned parameter was N_{top} , i.e. the number of the

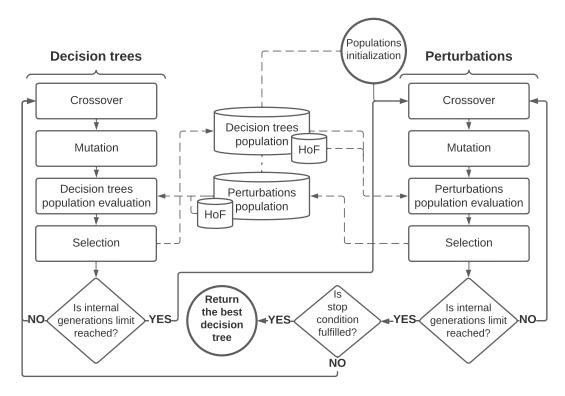


Figure 1: A high-level overview of the CoEvoDT algorithm.

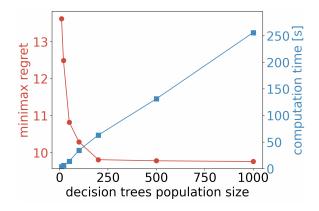


Figure 2: Comparison of minimax regret and computation size for **decision trees population size** values.

best individuals from the decision trees population involved in the perturbations evaluation. The results are presented in Figure 5 confirm the conjecture formulated in the main paper about the harmfulness of using the whole decision trees population ($N_{\rm top}=200$). Also, small values of this parameter ($N_{\rm top}<5$) lead to weaker results, A deeper examination of population structure in such cases reviles the presence of oscillations. In the extreme case of $N_{\rm top}=1$ (evaluation of a given perturbation is based on the best decision tree only) we observed that the perturbations population quickly losses diversity. Individuals in the population become similar to one another because they are optimized with respect to only one

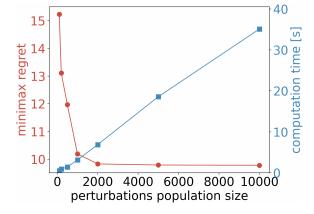


Figure 3: Comparison of minimax regret and computation size for **perturbations population size** values.

decision tree. As a result, the perturbations population returns a good response only to this particular decision tree, and in the next coevolution phase, the decision trees population is able to find with ease another solution for which there is no good response in the perturbations population. Afterwards, the whole perturbations population again adapts to the new best decision tree and "forgets" the previous ones. $N_{\rm top}=20$ appeared to be the best compromise between these two extremes (Figure 5).

dataset	CART	Meta Silvae	RIGDT-h	GROOT	FPRDT	CoEvoRDT	CoEvoRDT+FPRDT
ionos	0	2	1	1	1	2	3
breast	0	2	1	1	1	2	3
diabetes	0	2	1	1	1	3	4
bank	1	5	2	2	2	6	8
Japan3v4	1	8	3	3	3	9	12
spam	1	11	4	4	4	13	17
GesDvP	1	10	4	4	4	11	15
har1v2	1	11	4	4	4	12	15
wine	2	6	6	6	6	2	8
collision-det	9	26	16	18	16	17	35
mnist-1-5	8	14	8	8	8	13	20
mnist-2-6	6	19	8	8	7	24	31
mnist	17	62	22	21	21	68	93
f-mnist2v5	7	20	9	9	8	23	34
f-mnist3v4	7	23	10	9	9	25	36
f-mnist7v9	7	21	10	9	9	26	35
f-mnist	18	60	21	19	19	79	97
cifar10:0v5	21	112	42	39	40	146	191
cifar10:0v6	22	107	44	44	42	126	174
cifar10:4v8	21	106	41	41	41	111	163

Table 1: Comparison of methods' computation times (in seconds).

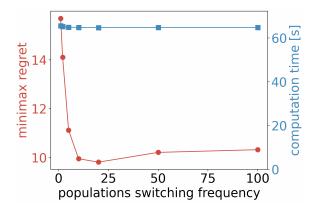


Figure 4: Comparison of minimax regret and computation size for **populations evaluation switching frequency** l_c).



Calculating the exact value of adversarial accuracy or minimax regret is not straightforward. It requires finding a perturbation that minimizes accuracy or maximizes regret from the infinite set of possible perturbations. Since this task is not trivial, we decided to estimate the real values of these metrics by drawing a uniformly random subset P of possible perturbations and then calculating the performance of all models on this subset.

In order to assess how large this subset should be to fairly estimate the performance of models, we chose 5 datasets with different ε values and ran each tested method on each dataset 5 times. This resulted in 25 decision trees. We then checked the following values for the size of P: $10^2, 10^3, 10^4, 10^5, 10^6, 10^7$. For each value of P, we drew

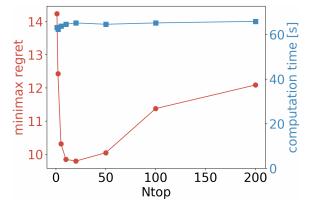


Figure 5: Comparison of minimax regret and computation size for the number of best individuals used to evaluate the perturbations ($N_{\rm top}$).

a given number of random perturbations and then evaluated all 25 decision trees using minimax regret and adversarial accuracy. The results for all models were then averaged. This procedure was repeated 20 times (each time a new subset of perturbations was drawn, but the 25 models remained the same) for each value of P. The mean value and standard error for the tested values of P are presented in Table 2. It shows that the standard error value decreases with the size of P. This is expected, as a larger subset of perturbations allows for a more thorough search of the space of possible perturbations and indicates that the results are becoming more reliable.

The standard error for small values ($\log_{10} |P| \le 4$) is high, which shows that the calculated metrics values are un-

reliable. However, for P sizes of at least 10^5 , the difference between multiple perturbations drawn is small and the results are stabilized. This does not indicate how close to the exact (real) values we are, but it does show that 10^5 is a large enough size of drawn perturbations sample to fairly assess and compare tested models.

	minima	ax regret	adversarial accuracy				
$log_{10} P $	average	std error	average	std error			
2	0.0964	0.0065	0.740	0.0088			
3	0.0968	0.0053	0.737	0.0056			
4	0.0972	0.0029	0.733	0.0036			
5	0.0976	0.0005	0.729	0.0006			
6	0.0977	0.0002	0.728	0.0005			
7	0.0977	0.0001	0.728	0.0003			

Table 2: Mean value and standard error of adversarial accuracy and minimax regret for different values of the size of random perturbations sample used to their calculation.

The role of Table 2 was to determine what perturbation set size should be used in experimental procedure from the main paper (Tables 2 and 3). Table 2 presented above explores how varying perturbation set size impacts results (variance arises from perturbation-related randomness), while the main paper's Tables 2 and 3 employ a different setup: for each method and each individual run, a new decision tree was generated, and subsequently, these generated decision trees were evaluated with a uniform and unchanging set of perturbations (with size of 10^5). In this context, the variance arises from the diversity among the decision tree models themselves.

Results for all datasets

Tables 3–21 show the computation time and the best outcomes in terms of minimax regret and adversarial accuracy resulting from multiple runs of FPRDT, CoEvoRDT, and CoEvoRDT initialized with various FPRDT outcomes. Obtained results confirm statements from the main paper that additional runs can significantly enhance the results for all methods. With regards to minimax regret FPRDT failed to approach a solution comparable to the individual outcome of CoEvoRDT. In terms of adversarial accuracy, multiple runs of FPRDT outperformed CoEvoRDT, yet as the number of iterations increased, both approaches appeared to converge towards similar outcomes.

Tables 22–40 display the outcomes across various sizes of the Hall of Fame (HoF). For each size, five different approaches for constructing the HoF are evaluated: including one *mixed tree* from the mixed Nash equilibrium after each generation (which serves as the baseline in CoEvoRDT), incorporating all *single trees* from the mixed Nash equilibrium, integrating only the single highest-fitness tree from the population, appending the top K highest-fitness individuals from the current population (with K representing the number of trees from the Nash mixed equilibrium), and adding one mixed tree composed of the top K highest-fitness trees with equal probabilities. Even a small HoF size significantly

enhances results for all the tested approaches. Furthermore, the strategy of introducing a Nash mixed tree appears to yield the most favorable outcomes, whereas relying solely on the addition of the single highest-fitness tree yields the least favorable results.

References

Uzilov, A. V.; Keegan, J. M.; and Mathews, D. H. 2006. Detection of non-coding RNAs on the basis of predicted secondary structure formation free energy change. *BMC bioinformatics*, 7(1): 1–30.

		minim	ax regret			adversari	ial accuracy			computat	ion time [s]	
NI	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	M EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	M EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
11	N FFKDI	N COEVORDI	+ N FPRDT	+ N FPRDT	N FFKD1	IN COEVORDT	+ N FPRDT	+ N FPRDT	N FFKD1	IN COEVORDT	+ N FPRDT	+ N FPRDT
1	.061	.052	.052	.052	.795	.791	.795	.795	1	2	3	3
2	.060	.051	.052	.051	.800	.797	.802	.803	2	4	4	5
3	.060	.051	.051	.051	.802	.805	.803	.811	3	6	5	10
4	.060	.051	.051	.051	.810	.812	.814	.818	4	9	8	12
5	.059	.051	.051	.051	.821	.827	.823	.828	5	11	11	16
10	.058	.050	.050	.050	.842	.836	.835	.840	10	19	12	28
20	.056	.048	.049	.048	.850	.849	.848	.850	20	37	22	57
50	.055	.048	.048	.048	.851	.853	.851	.852	49	104	40	164
100	.055	.048	.049	.048	.853	.858	.852	.857	99	192	109	275

Table 3: Best results of repeated N algorithms' runs for **ionos** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computation time [s]				
N	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT		
1	.057	.049	.049	.049	.876	.885	.889	.889	1	2	3	3		
2	.056	.049	.049	.049	.881	.893	.892	.896	2	4	4	6		
3	.056	.049	.048	.048	.887	.896	.902	.911	3	6	6	10		
4	.056	.048	.048	.048	.892	.917	.908	.916	4	8	8	12		
5	.056	.048	.048	.048	.909	.922	.924	.927	5	10	11	15		
10	.054	.048	.047	.047	.921	.936	.936	.940	10	18	12	28		
20	.052	.046	.046	.046	.935	.946	.946	.949	21	37	19	61		
50	.051	.046	.046	.045	.940	.955	.953	.949	51	102	39	161		
100	.051	.045	.046	.045	.938	.954	.947	.957	95	210	103	297		

Table 4: Best results of repeated N algorithms' runs for **breast** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT		CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT		N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.117	.096	.094	.094	.648	.617	.648	.648	1	3	4	4
2	.115	.096	.093	.093	.653	.622	.656	.657	2	6	5	9
3	.115	.095	.093	.093	.657	.626	.654	.663	3	9	7	13
4	.114	.095	.093	.093	.662	.633	.660	.663	4	12	12	16
5	.114	.094	.093	.092	.670	.642	.673	.673	5	16	15	21
10	.111	.092	.091	.090	.687	.655	.684	.684	11	30	17	39
20	.107	.090	.088	.088	.690	.661	.692	.691	21	64	29	88
50	.105	.089	.088	.087	.695	.667	.694	.694	50	158	62	202
100	.105	.089	.088	.087	.694	.667	.695	.692	105	314	154	426

Table 5: Best results of repeated N algorithms' runs for **diabetes** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT		CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.089	.076	.076	.076	.658	.657	.663	.663	2	6	8	8
2	.088	.076	.075	.075	.659	.663	.669	.668	4	12	11	15
3	.088	.075	.076	.075	.664	.669	.671	.678	6	17	14	22
4	.087	.075	.075	.075	.673	.674	.677	.683	9	22	21	33
5	.087	.075	.075	.075	.679	.686	.685	.687	10	28	29	40
10	.085	.074	.073	.073	.697	.694	.694	.698	21	62	32	83
20	.080	.071	.071	.071	.703	.705	.707	.706	42	115	66	154
50	.080	.071	.071	.071	.709	.705	.709	.711	101	318	131	450
100	.080	.070	.071	.070	.707	.713	.711	.709	206	650	291	922

Table 6: Best results of repeated N algorithms' runs for **bank** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	N EDDINT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
11	NITKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT	NITKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT	NITKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT
1	.066	.062	.061	.061	.667	.665	.668	.668	3	9	12	12
2	.066	.062	.061	.061	.671	.668	.673	.675	6	17	15	24
3	.065	.061	.060	.060	.673	.676	.678	.678	8	26	24	32
4	.064	.061	.060	.060	.685	.682	.685	.683	12	36	33	51
5	.064	.061	.060	.060	.692	.692	.691	.695	16	45	45	57
10	.063	.060	.059	.059	.706	.703	.700	.703	30	90	51	108
20	.060	.058	.057	.057	.710	.710	.712	.714	55	188	92	261
50	.059	.057	.057	.057	.719	.715	.713	.713	147	420	184	596
100	.059	.057	.057	.056	.716	.716	.715	.715	277	976	438	1256

Table 7: Best results of repeated N algorithms' runs for **Japan3v4** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.074	.070	.069	.069	.746	.751	.753	.753	4	13	17	17
2	.073	.070	.069	.068	.748	.760	.756	.757	8	28	20	37
3	.073	.069	.069	.068	.752	.760	.764	.765	12	38	36	54
4	.072	.069	.068	.068	.762	.775	.768	.777	16	55	47	66
5	.072	.069	.068	.068	.773	.788	.777	.781	18	62	69	78
10	.070	.068	.067	.066	.783	.795	.793	.796	40	127	71	183
20	.067	.066	.065	.064	.794	.806	.800	.801	76	255	134	331
50	.066	.065	.065	.064	.801	.807	.806	.803	218	627	261	795
100	.066	.065	.064	.064	.803	.815	.808	.811	394	1193	581	1722

Table 8: Best results of repeated N algorithms' runs for **spam** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT	N CORVORINI	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.131	.114	.114	.114	.735	.740	.741	.741	4	11	15	15
2	.130	.114	.113	.114	.737	.746	.745	.748	8	21	19	29
3	.129	.113	.113	.113	.740	.751	.749	.759	12	34	29	42
4	.128	.112	.112	.112	.754	.766	.757	.759	16	46	43	55
5	.127	.112	.112	.112	.761	.771	.770	.768	19	57	55	71
10	.124	.110	.111	.110	.772	.782	.781	.785	43	115	71	151
20	.119	.107	.107	.106	.784	.793	.793	.790	80	198	134	267
50	.117	.106	.106	.106	.791	.795	.788	.791	205	540	194	748
100	.117	.106	.106	.106	.788	.801	.796	.798	399	1159	535	1441

Table 9: Best results of repeated N algorithms' runs for **GesDvP** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.068	.064	.064	.064	.804	.818	.820	.820	4	12	15	15
2	.067	.064	.064	.063	.808	.828	.823	.828	8	22	21	31
3	.067	.063	.063	.063	.817	.832	.828	.839	12	36	28	53
4	.067	.063	.063	.063	.824	.839	.840	.844	16	52	42	74
5	.066	.063	.063	.063	.835	.853	.847	.852	19	55	63	73
10	.065	.062	.062	.062	.848	.868	.866	.866	38	122	65	153
20	.062	.060	.060	.060	.858	.875	.876	.878	84	239	136	317
50	.061	.059	.059	.059	.868	.877	.880	.880	200	608	272	852
100	.061	.060	.059	.059	.869	.887	.882	.881	403	1253	656	1493

Table 10: Best results of repeated N algorithms' runs for **har1v2** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N EDDDT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
11	N FFKDI	N COEVORDI	+ N FPRDT	+ N FPRDT	N FFKD1	IN COEVORDT	+ N FPRDT	+ N FPRDT	N FFKDI	IN COEVORD I	+ N FPRDT	+ N FPRDT
1	.109	.090	.090	.090	.674	.688	.692	.692	6	2	8	8
2	.108	.089	.089	.089	.675	.692	.696	.699	11	4	13	15
3	.107	.089	.089	.089	.682	.699	.702	.707	19	6	16	23
4	.107	.089	.088	.088	.686	.707	.707	.709	26	8	15	33
5	.107	.088	.089	.089	.699	.715	.715	.721	30	11	19	38
10	.103	.087	.087	.087	.713	.725	.731	.728	63	21	21	86
20	.099	.084	.084	.084	.724	.736	.739	.743	119	36	32	148
50	.098	.084	.084	.084	.722	.740	.741	.738	295	98	46	433
100	.098	.083	.084	.084	.723	.741	.737	.745	599	197	107	716

Table 11: Best results of repeated N algorithms' runs for **wine** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	M EDDINT			N CoEvoRDT	N EDDINT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
11	NTIKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT	NITKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT	NITKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT
1	.091	.061	.059	.059	.792	.798	.803	.803	16	17	35	35
2	.091	.061	.059	.058	.793	.801	.813	.809	33	31	46	69
3	.090	.061	.058	.059	.801	.806	.818	.816	47	46	65	96
4	.090	.060	.058	.058	.810	.821	.825	.823	63	74	80	141
5	.089	.060	.058	.058	.817	.834	.829	.831	79	89	105	165
10	.087	.059	.057	.057	.832	.845	.847	.849	151	171	110	299
20	.082	.057	.055	.055	.845	.852	.852	.862	304	360	208	619
50	.082	.057	.055	.055	.852	.862	.855	.864	847	919	396	1637
100	.081	.057	.055	.055	.852	.867	.861	.861	1643	1685	897	3335

Table 12: Best results of repeated N algorithms' runs for **collision-det** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	M EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	M EDDDT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
111	N FFKDI	IN COEVORDI	+ N FPRDT	+ N FPRDT	N FFKDI	IN COEVORDT	+ N FPRDT	+ N FPRDT	N FFKDI	IN COEVORD I	+ N FPRDT	+ N FPRDT
1	.067	.055	.055	.055	.966	.964	.969	.969	8	13	20	20
2	.066	.054	.055	.055	.968	.966	.971	.971	15	26	30	40
3	.066	.054	.055	.055	.968	.968	.974	.975	23	41	44	60
4	.065	.054	.054	.054	.972	.971	.975	.976	33	47	52	85
5	.065	.054	.054	.054	.974	.975	.977	.977	43	63	61	113
10	.064	.053	.053	.053	.980	.978	.982	.983	72	130	74	190
20	.061	.052	.052	.051	.984	.980	.986	.987	161	237	131	418
50	.060	.051	.052	.051	.984	.982	.986	.985	395	605	248	1041
100	.060	.051	.052	.051	.984	.984	.985	.987	770	1272	609	2095

Table 13: Best results of repeated N algorithms' runs for **mnist:1v5** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N EDDLAT	N CoEvoRDT		N CoEvoRDT	N EDD DT	N CoEvoRDT		N CoEvoRDT	N EDD DT	N CoEvoRDT		N CoEvoRDT
14	NTIKDI	IN COLVORDI	+ N FPRDT	+ N FPRDT	NTIKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT	NTIKDI	IN COLVORD I	+ N FPRDT	+ N FPRDT
1	.069	.055	.054	.054	.922	.917	.922	.922	7	24	31	31
2	.068	.055	.054	.054	.927	.919	.928	.929	15	43	40	62
3	.068	.054	.053	.053	.932	.925	.932	.934	19	70	44	102
4	.068	.054	.053	.053	.942	.939	.943	.944	30	91	59	115
5	.067	.054	.053	.053	.953	.947	.954	.956	37	123	69	141
10	.066	.053	.052	.052	.961	.954	.963	.963	68	219	98	280
20	.063	.051	.051	.050	.965	.958	.965	.967	129	486	145	618
50	.062	.051	.050	.050	.969	.963	.969	.970	349	1311	397	1669
100	.062	.051	.050	.050	.971	.969	.972	.973	680	2385	675	3281

Table 14: Best results of repeated N algorithms' runs for **mnist:2v6** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	ial accuracy			computat	ion time [s]	
NI	M EDDDT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	M EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	M EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
11	N FFKDI	IN COEVORDT	+ N FPRDT	+ N FPRDT	N FFKD1	IN COEVORDT	+ N FPRDT	+ N FPRDT	N FFKDI	IN COEVORD I	+ N FPRDT	+ N FPRDT
1	.124	.113	.112	.112	.742	.745	.754	.754	21	68	93	93
2	.123	.112	.112	.111	.746	.751	.757	.760	45	144	120	177
3	.122	.112	.111	.111	.749	.752	.768	.772	62	211	191	280
4	.122	.111	.111	.111	.761	.769	.770	.778	81	286	236	356
5	.121	.111	.110	.110	.764	.780	.780	.784	112	345	310	421
10	.118	.109	.108	.108	.782	.789	.792	.798	213	745	394	878
20	.112	.106	.105	.104	.792	.798	.806	.809	410	1252	784	1790
50	.112	.105	.104	.104	.800	.802	.807	.805	973	3138	1341	4361
100	.111	.105	.105	.104	.795	.807	.806	.810	2210	6755	3241	8561

Table 15: Best results of repeated N algorithms' runs for **mnist** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	ial accuracy			computat	ion time [s]	
N	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.238	.196	.196	.196	.978	.982	.982	.982	8	23	34	34
2	.236	.195	.195	.196	.979	.983	.984	.984	17	51	40	61
3	.234	.195	.195	.195	.982	.984	.984	.984	26	66	71	84
4	.233	.194	.193	.192	.982	.984	.984	.985	32	91	80	114
5	.232	.192	.193	.192	.982	.984	.985	.985	37	116	98	153
10	.227	.189	.189	.189	.983	.985	.985	.987	87	220	145	282
20	.215	.183	.183	.183	.983	.985	.987	.987	176	479	258	716
50	.215	.183	.182	.182	.984	.987	.987	.987	431	1037	475	1527
100	.214	.181	.183	.183	.985	.987	.987	.988	843	2496	1091	3381

Table 16: Best results of repeated N algorithms' runs for **F-mnist2v5** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT	N Colivority	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT		N CoEvoRDT + N FPRDT	N FPRDT	N CoEvoRDT	CoEvoRDT + N FPRDT	N CoEvoRDT + N FPRDT
1	.232	.202	.199	.199	.865	.869	.870	.870	9	25	36	36
2	.230	.201	.197	.198	.874	.873	.875	.882	16	50	45	61
3	.229	.201	.197	.197	.877	.885	.880	.884	25	68	65	90
4	.227	.199	.197	.196	.887	.892	.893	.890	39	97	82	135
5	.226	.199	.196	.196	.891	.906	.900	.904	43	131	117	158
10	.222	.196	.192	.191	.915	.920	.913	.922	81	261	157	368
20	.210	.189	.186	.185	.927	.930	.929	.930	163	452	298	631
50	.208	.187	.186	.185	.929	.932	.928	.933	474	1235	498	1739
100	.209	.187	.186	.185	.933	.939	.927	.937	870	2748	1302	3919

Table 17: Best results of repeated N algorithms' runs for **F-mnist3v4** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N EDD DT			N CoEvoRDT	N EDDINT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	N EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
11	NITKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT	NITKDI	IN COLVORDT	+ N FPRDT	+ N FPRDT	NITKDI	IN COLVORD I	+ N FPRDT	+ N FPRDT
1	.240	.208	.207	.207	.876	.868	.880	.880	9	26	35	35
2	.238	.206	.205	.206	.885	.878	.890	.885	19	51	47	71
3	.235	.205	.204	.204	.886	.884	.888	.897	28	76	74	100
4	.235	.206	.204	.204	.892	.896	.903	.905	37	95	95	126
5	.234	.205	.203	.204	.905	.911	.909	.918	42	129	118	161
10	.227	.200	.201	.200	.920	.916	.924	.929	83	245	138	355
20	.217	.195	.193	.192	.933	.933	.937	.937	166	501	286	658
50	.216	.192	.194	.193	.938	.934	.942	.945	430	1312	535	1657
100	.216	.192	.193	.192	.942	.942	.944	.939	878	2825	1270	3528

Table 18: Best results of repeated N algorithms' runs for **F-mnist7v9** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT	N CoEvoRDT		N CoEvoRDT	N FPRDT	N CoEvoRDT		N CoEvoRDT	N FPRDT	N CoEvoRDT		N CoEvoRDT
11	TTT TO	T COL TOTOL	+ N FPRDT	+ N FPRDT	TTT REF	T COL TOTOL	+ N FPRDT	+ N FPRDT	TTT KDT	TO COLVORD I	+ N FPRDT	+ N FPRDT
1	.314	.241	.238	.238	.678	.685	.693	.692	40	146	191	178
2	.310	.239	.236	.235	.683	.695	.702	.704	82	282	230	338
3	.308	.237	.234	.234	.688	.699	.711	.719	120	429	271	534
4	.306	.236	.233	.233	.699	.714	.718	.723	158	599	297	747
5	.304	.235	.232	.232	.714	.732	.734	.735	199	730	347	865
10	.294	.229	.226	.225	.745	.746	.753	.759	406	1478	560	1729
20	.276	.218	.215	.214	.764	.765	.772	.774	808	2926	956	3513
50	.272	.216	.214	.214	.766	.768	.772	.775	2095	7662	2234	9278
100	.272	.216	.214	.214	.766	-	.774	-	4008	>10000	4179	>10000

Table 19: Best results of repeated N algorithms' runs for **cifar10:0v5** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
NI	M EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	M EDDINT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT	M EDD DT	N CoEvoRDT	CoEvoRDT	N CoEvoRDT
11	N FFKDI	IN COEVORDI	+ N FPRDT	+ N FPRDT	N FFKDI	IN COEVORDT	+ N FPRDT	+ N FPRDT	NFFKDI	IN COEVORD I	+ N FPRDT	+ N FPRDT
1	.341	.289	.289	.289	.688	.692	.697	.697	42	126	174	174
2	.338	.286	.287	.287	.692	.694	.705	.704	77	255	200	336
3	.336	.286	.285	.286	.698	.700	.710	.709	130	364	315	497
4	.336	.284	.286	.285	.702	.712	.716	.714	163	455	450	577
5	.334	.285	.284	.283	.709	.725	.720	.722	218	629	552	804
10	.324	.279	.278	.280	.727	.731	.730	.738	453	1366	671	1865
20	.309	.270	.271	.271	.737	.744	.743	.742	910	2639	1503	3563
50	.307	.268	.269	.270	.738	.744	.745	.744	2003	6210	2498	7412
100	.306	.267	.270	.267	.742	-	.749	-	3897	>10000	6155	>10000

Table 20: Best results of repeated N algorithms' runs for **cifar10:0v6** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minim	ax regret			adversari	al accuracy			computat	ion time [s]	
N	N FPRDT	N CoEvoRDT		N CoEvoRDT	N FPRDT	N CoEvoRDT		N CoEvoRDT	N FPRDT	N CoEvoRDT		N CoEvoRDT
			+ N FPRDT	+ N FPRDT			+ N FPRDT	+ N FPRDT			+ N FPRDT	+ N FPRDT
1	.331	.283	.281	.281	.661	.663	.664	.664	41	111	163	163
2	.327	.280	.279	.280	.665	.667	.669	.668	74	209	202	289
3	.327	.281	.279	.279	.669	.672	.675	.675	125	306	291	397
4	.324	.279	.278	.278	.678	.683	.679	.682	180	403	356	604
5	.324	.279	.277	.277	.684	.691	.690	.692	218	523	446	799
10	.315	.273	.273	.271	.697	.699	.699	.705	436	1158	625	1699
20	.301	.263	.264	.263	.709	.713	.708	.713	888	2399	1131	3237
50	.296	.263	.261	.260	.710	.713	.708	.708	1888	5628	2314	8033
100	.298	.261	.263	.261	.714	-	.711	-	4039	>10000	6193	>10000

Table 21: Best results of repeated N algorithms' runs for **cifar10:4v8** dataset. In CoEvoRDT + N FPRDT resulting DTs from N independent runs of FPRDT were incorporated into CoEvoRDT initial population.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	10p K	Dest
0	.058	.058	.058	.058	.059	.778	.778	.778	.778	.778	0.6	0.6	0.6	0.6	0.6
10	.053	.055	.056	.056	.057	.786	.784	.783	.781	.780	0.7	0.6	0.6	0.6	0.6
20	.053	.055	.055	.055	.057	.786	.785	.784	.782	.781	0.7	0.7	0.7	0.7	0.7
50	.053	.054	.055	.055	.056	.787	.785	.784	.783	.782	0.8	0.7	0.7	0.8	0.7
100	.052	.053	.054	.054	.056	.789	.787	.786	.784	.782	0.9	0.8	0.8	0.9	0.7
200	.052	.053	.054	.054	.055		.788	.787	.784	.782	1.0	0.9	0.9	1.0	0.9
500	.052	.053	.054	.054	.055	.791	.789	.788	.785	.782	1.2	1.0	1.0	1.2	0.9
∞	.052	.052	.054	.053	.054	.792	.790	.789	.787	.783	1.2	1.0	1.0	1.2	0.9

Table 22: Results with respect of HoF size for **ionos** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	тор к	Dest
0	.054	.055	.055	.055	.055	.870	.870	.870	.870	.870	1.2	1.2	1.2	1.2	1.2
10	.050	.051	.052	.052	.054	.879	.878	.876	.874	.873	1.4	1.3	1.3	1.2	1.2
20	.050	.051	.052	.052	.053	.880	.878	.877	.875	.874	1.6	1.3	1.3	1.5	1.4
50	.050	.051	.052	.052	.053	.880	.879	.877	.876	.875	1.7	1.5	1.3	1.6	1.4
100	.049	.050	.051	.051	.052	.883	.881	.879	.877	.875	1.6	1.5	1.5	1.7	1.4
200	.049	.050	.051	.051	.052	.885	.882	.881	.877	.875	2.0	1.8	1.8	2.1	1.8
500	.049	.050	.051	.051	.052	.885	.883	.881	.878	.875	2.3	2.1	2.2	2.5	2.0
Inf	.049	.049	.051	.050	.051	.886	.883	.883	.880	.876	2.3	2.2	2.2	2.6	2.1

Table 23: Results with respect of HoF size for **breast** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	Top K	Dest
0	.103	.108	.108	.108	.108	.607	.607	.607	.607	.607	1.6	1.6	1.6	1.6	1.6
10	.098	.101	.103	.102	.106	.613	.612	.611	.609	.608	2.0	1.9	1.8	1.9	1.8
20	.098	.101	.102	.102	.104	.613	.612	.611	.610	.609	2.2	2.1	1.8	2.0	1.9
50	.097	.099	.101	.101	.104	.614	.613	.611	.611	.610	2.6	2.1	2.0	2.5	2.0
100	.096	.098	.100	.100	.103	.615	.614	.613	.611	.610	2.5	2.6	2.2	2.6	2.4
200	.096	.098	.099	.099	.101	.617	.615	.614	.612	.610	3.0	2.8	2.5	2.9	2.7
500	.096	.098	.099	.099	.101	.617	.615	.615	.612	.610	3.5	3.3	3.2	3.3	2.6
∞	.095	.096	.099	.097	.100	.618	.616	.615	.614	.611	3.3	3.4	3.2	3.6	3.0

Table 24: Results with respect of HoF size for **diabetes** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	TOP K	Dest
0	.083	.085	.085	.085	.086	.646	.646	.646	.646	.646	3.3	3.3	3.3	3.3	3.3
10	.078	.080	.081	.081	.084	.653	.651	.650	.649	.648	4.1	3.7	3.8	3.6	3.9
20	.078	.080	.080	.081	.083	.653	.652	.651	.649	.649	4.2	3.7	4.0	4.7	4.0
50	.077	.079	.080	.080	.082	.654	.652	.651	.650	.649	5.4	4.8	4.1	5.0	4.0
100	.076	.078	.079	.079	.081	.655	.654	.653	.651	.649	5.3	4.5	4.2	5.0	4.6
200	.076	.077	.079	.079	.080	.657	.655	.654	.651	.650	6.0	5.0	5.4	6.7	5.0
500	.076	.077	.079	.078	.080	.657	.655	.654	.652	.649	6.5	6.5	6.8	7.1	5.1
∞	.075	.076	.078	.077	.079	.658	.656	.655	.654	.650	7.5	6.8	6.8	8.0	5.6

Table 25: Results with respect of HoF size for **bank** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Ract	Nash mixed	Top K as	Nash single	Тор К	Rect
nor size	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	торк	Desi	tree	mixed tree	trees	TOP K	Dest
0	.066	.070	.070	.070	.070	.654	.654	.654	.654	.654	5.2	5.2	5.2	5.2	5.2
10	.063	.065	.066	.066	.068	.661	.659	.658	.657	.656	5.9	5.9	5.1	6.0	5.5
20	.063	.065	.066	.066	.067	.661	.660	.659	.657	.657	6.0	6.6	6.3	6.3	5.9
50	.063	.064	.066	.065	.067	.662	.660	.659	.658	.657	7.8	6.9	6.3	7.5	5.9
100	.062	.064	.065	.065	.066	.663	.662	.661	.659	.657	7.4	7.5	6.6	7.4	7.2
200	.062	.063	.064	.064	.066	.665	.663	.662	.659	.658	9.1	8.9	7.6	10.0	7.9
500	.062	.063	.064	.064	.065	.665	.663	.662	.660	.657	10.5	9.6	10.0	12.0	8.9
∞	.062	.062	.064	.063	.065	.666	.664	.663	.662	.658	10.0	9.6	10.2	12.1	9.0

Table 26: Results with respect of HoF size for **Japan3v4** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	10p K	Dest
0	.081	.079	.079	.078	.079	.739	.739	.739	.739	.739	7.1	7.1	7.1	7.1	7.1
10	.072	.073	.075	.075	.077	.746	.745	.743	.742	.741	8.3	8.4	8.1	9.1	7.7
20	.072	.073	.074	.074	.076	.747	.745	.744	.742	.741	9.5	9.7	9.0	10.2	9.7
50	.071	.073	.074	.074	.076	.747	.746	.744	.743	.742	11.0	10.4	8.9	9.7	10.1
100	.070	.072	.073	.073	.075	.749	.747	.746	.744	.742	11.5	10.1	10.1	11.7	10.4
200	.070	.071	.073	.072	.074	.751	.749	.747	.745	.743	13.2	12.7	10.8	14.3	11.1
500	.070	.071	.072	.072	.074	.751	.749	.748	.745	.742	15.7	13.5	13.7	15.8	12.5
∞	.070	.070	.072	.071	.073	.752	.750	.749	.747	.743	17.1	13.5	13.9	15.9	12.8

Table 27: Results with respect of HoF size for **spam** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	10p K	Dest	tree	mixed tree	trees	Top K	Dest
0	.122	.128	.128	.128	.128	.728	.728	.728	.728	.728	6.1	6.1	6.1	6.1	6.1
10	.117	.120	.122	.122	.125	.735	.734	.732	.731	.730	6.9	7.8	7.3	8.0	6.3
20	.117	.120	.121	.121	.124	.736	.734	.733	.731	.731	8.5	7.9	7.2	9.2	8.3
50	.115	.118	.120	.120	.123	.736	.735	.733	.732	.731	8.8	8.2	8.2	9.8	7.4
100	.114	.117	.119	.119	.122	.738	.737	.735	.733	.731	10.6	9.6	8.2	10.2	8.7
200	.114	.116	.118	.118	.120	.740	.738	.736	.734	.732	11.5	10.1	11.0	12.0	10.5
500	.114	.116	.118	.118	.120	.740	.738	.737	.734	.731	12.7	12.3	12.4	14.2	11.8
∞	.113	.114	.118	.116	.119	.741	.739	.738	.736	.732	13.9	12.6	12.5	14.3	12.2

Table 28: Results with respect of HoF size for **GesDvP** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Doct	Nash mixed	Top K as	Nash single	Ton V	Doct	Nash mixed	Top K as	Nash single	Тор К	Post
HOF SIZE	tree	mixed tree	trees	Top K	Dest	tree	mixed tree	trees	Top K	Desi	tree	mixed tree	trees	10p K	Dest
0	.075	.072	.072	.072	.072	.804	.804	.804	.804	.804	7.0	7.0	7.0	7.0	7.0
10	.065	.067	.068	.068	.070	.812	.811	.810	.808	.807	7.8	8.0	7.7	7.4	8.0
20	.065	.067	.068	.068	.070	.813	.812	.811	.809	.808	8.6	7.7	7.8	9.1	7.6
50	.065	.066	.068	.068	.069	.814	.812	.811	.810	.808	9.4	8.8	9.3	8.7	8.4
100	.064	.066	.067	.067	.069	.816	.814	.813	.811	.808	11.6	9.7	8.9	10.2	9.7
200	.064	.065	.066	.066	.068	.818	.815	.814	.811	.809	12.1	10.2	9.9	11.3	11.1
500	.064	.065	.066	.066	.068	.818	.816	.815	.812	.809	15.4	12.3	13.1	15.4	10.5
∞	.064	.064	.066	.065	.067	.819	.817	.816	.814	.809	14.1	12.4	13.2	15.7	12.5

Table 29: Results with respect of HoF size for har1v2 dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	10p K	Dest
0	.105	.101	.101	.101	.101	.677	.677	.677	.677	.677	1.3	1.3	1.3	1.3	1.3
10	.092	.094	.096	.096	.099	.683	.682	.681	.679	.678	1.4	1.5	1.3	1.4	1.5
20	.092	.094	.095	.095	.098	.684	.683	.682	.680	.679	1.6	1.4	1.5	1.7	1.6
50	.091	.093	.095	.095	.097	.684	.683	.682	.681	.680	1.8	1.6	1.6	1.8	1.6
100	.090	.092	.094	.094	.096	.686	.685	.683	.682	.680	2.0	1.7	1.9	1.9	1.5
200	.090	.092	.093	.093	.095	.688	.686	.685	.682	.681	2.3	1.9	2.0	2.3	1.9
500	.090	.092	.093	.093	.095	.688	.686	.685	.683	.680	2.5	2.3	2.2	2.9	2.1
∞	.089	.090	.093	.091	.094	.689	.687	.686	.684	.681	2.9	2.5	2.4	3.0	2.2

Table 30: Results with respect of HoF size for **wine** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Ract	Nash mixed	Top K as	Nash single	Тор К	Rect
nor size	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	торк	Desi	tree	mixed tree	trees	Top K	Dest
0	.066	.068	.068	.068	.069	.785	.785	.785	.785	.785	9.8	9.8	9.8	9.8	9.8
10	.062	.064	.065	.065	.067	.793	.791	.790	.788	.787	10.1	9.9	10.7	11.5	9.4
20	.062	.064	.065	.065	.066	.793	.792	.791	.789	.788	12.5	12.2	11.8	13.0	11.5
50	.062	.063	.064	.064	.066	.794	.792	.791	.790	.789	12.8	11.9	11.2	11.9	11.7
100	.061	.063	.064	.064	.065	.796	.794	.793	.791	.789	14.4	14.5	12.3	13.7	12.6
200	.061	.062	.063	.063	.064	.798	.795	.794	.791	.789	16.8	16.6	14.4	16.3	15.2
500	.061	.062	.063	.063	.064	.798	.796	.795	.792	.789	17.9	17.0	15.7	18.7	14.7
∞	.061	.061	.063	.062	.064	.799	.797	.796	.794	.790	18.6	17.3	18.2	18.9	16.6

Table 31: Results with respect of HoF size for **collision-det** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	x regret					al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
nor size	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	торк	Dest
0	.062	.062	.062	.062	.062	.948	.948	.948	.948	.948	9.6	9.6	9.6	9.6	9.6
10	.056	.058	.059	.059	.060	.958	.956	.954	.952	.951	12.3	10.3	11.4	10.8	10.5
20	.056	.058	.058	.058	.060	.958	.957	.955	.953	.952	13.5	11.5	10.7	12.4	12.0
50	.056	.057	.058	.058	.060	.959	.957	.955	.954	.953	15.6	12.6	12.6	13.1	11.2
100	.055	.056	.057	.057	.059	.961	.959	.958	.955	.953	17.3	15.3	14.6	14.0	13.0
200	.055	.056	.057	.057	.058	.964	.961	.959	.956	.954	17.5	16.0	14.4	17.8	16.7
500	.055	.056	.057	.057	.058	.964	.962	.960	.956	.953	21.5	17.0	19.2	21.0	14.9
∞	.055	.055	.057	.056	.057	.965	.962	.961	.959	.954	19.6	18.6	19.8	21.2	17.0

Table 32: Results with respect of HoF size for **mnist:v1-5** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computat	ion time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Ract	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Best
HOF SIZE	tree	mixed tree	trees	Top K	Dest	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	TOP K	Best
0	.062	.062	.062	.062	.062	.912	.912	.912	.912	.912	81.2	81.2	81.2	81.2	81.2
10	.056	.057	.057	.058	.061	.915	.915	.914	.914	.912	91.1	84.7	87.8	86.0	84.0
20	.057	.058	.057	.058	.060	.915	.915	.915	.915	.913	95.0	88.1	97.5	97.8	91.2
50	.055	.057	.057	.058	.060	.916	.916	.915	.915	.912	116.2	91.5	104.4	100.5	102.2
100	.055	.056	.056	.057	.059	.917	.918	.917	.917	.914	125.1	94.7	123.3	114.7	110.6
200	.055	.055	.055	.056	.058	.917	.917	.916	.916	.914	132.3	102.1	132.6	139.0	124.1
500	.054	.055	.055	.056	.058	.917	.917	.917	.916	.915	154.2	114.5	163.2	173.8	144.9
∞	.054	.055	.055	.055	.058	.917	.917	.917	.917	.915	159.4	134.0	179.5	177.2	153.0

Table 33: Results with respect of HoF size for **mnist:2v6** dataset. ∞ means that there was no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Ract	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
TIOI SIZE	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	торк	Dest
0	.121	.127	.127	.127	.127	.733	.733	.733	.733	.733	38.7	38.7	38.7	38.7	38.7
10	.116	.119	.121	.121	.124	.740	.739	.737	.736	.735	44.0	41.7	39.0	43.1	41.5
20	.116	.118	.120	.120	.123	.741	.739	.738	.736	.735	49.6	45.3	47.9	48.8	46.5
50	.114	.117	.119	.119	.122	.741	.740	.738	.737	.736	51.6	52.6	51.3	49.6	49.4
100	.113	.116	.118	.118	.121	.743	.742	.740	.738	.736	67.1	53.7	50.1	64.7	55.2
200	.113	.115	.117	.117	.119	.745	.743	.741	.739	.737	68.1	66.9	54.3	73.5	60.6
500	.113	.115	.117	.117	.119	.745	.743	.742	.739	.736	74.3	68.5	70.7	79.8	64.5
∞	.112	.113	.117	.115	.118	.746	.744	.743	.741	.737	82.1	69.2	71.1	80.5	70.8

Table 34: Results with respect of HoF size for **mnist** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
HOF SIZE	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	Top K	Dest
0	.226	.220	.220	.220	.221	.966	.966	.966	.966	.966	15.0	15.0	15.0	15.0	15.0
10	.200	.206	.210	.209	.215	.975	.974	.972	.970	.968	15.3	13.8	14.6	16.0	13.2
20	.200	.206	.208	.208	.213	.976	.974	.973	.971	.969	15.5	17.0	14.8	17.9	14.9
50	.198	.203	.207	.207	.212	.977	.975	.973	.972	.970	20.2	18.9	16.6	17.1	17.2
100	.196	.201	.205	.205	.210	.979	.977	.975	.973	.970	20.6	19.9	19.7	20.8	17.2
200	.196	.200	.203	.203	.207	.982	.979	.977	.974	.971	23.7	20.3	22.0	23.4	22.4
500	.196	.199	.203	.202	.207	.982	.980	.978	.974	.971	26.4	25.8	22.2	28.3	21.3
∞	.195	.197	.202	.199	.205	.983	.980	.979	.977	.972	27.0	25.9	25.3	28.9	21.7

Table 35: Results with respect of HoF size for **f-mnist:2v5** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Doct	Nash mixed	Top K as	Nash single	Ton V	Doct	Nash mixed	Top K as	Nash single	Тор К	Doct
HOF SIZE	tree	mixed tree	trees	Top K	Dest	tree	mixed tree	trees	Top K	Desi	tree	mixed tree	trees	10p K	Dest
0	.221	.227	.227	.226	.227	.855	.855	.855	.855	.855	13.9	13.9	13.9	13.9	13.9
10	.206	.212	.216	.216	.222	.863	.862	.860	.858	.857	16.0	15.7	16.1	15.9	16.3
20	.207	.212	.214	.214	.220	.864	.862	.861	.859	.858	18.8	16.1	15.1	18.8	16.4
50	.204	.209	.213	.213	.219	.864	.863	.861	.860	.859	21.7	20.1	17.9	18.4	19.4
100	.202	.207	.211	.211	.216	.867	.865	.863	.861	.859	23.3	19.9	18.5	21.7	20.4
200	.202	.206	.209	.209	.213	.869	.866	.865	.862	.860	25.2	23.2	20.1	24.9	23.4
500	.202	.206	.209	.209	.213	.869	.867	.866	.862	.859	28.6	24.2	26.1	27.6	25.5
∞	.201	.203	.208	.205	.211	.870	.867	.867	.864	.860	30.8	25.8	26.3	28.6	25.9

Table 36: Results with respect of HoF size for **f-mnist:3v4** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret				adversaria	al accuracy				computati	on time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
TIOI SIZE	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	торк	Dest
0	.227	.233	.233	.233	.234	.854	.854	.854	.854	.854	15.1	15.1	15.1	15.1	15.1
10	.213	.218	.222	.222	.229	.862	.861	.859	.857	.856	18.4	15.9	14.3	16.3	16.9
20	.213	.218	.220	.221	.226	.863	.861	.860	.858	.857	18.1	17.1	17.9	19.7	17.5
50	.210	.216	.220	.220	.225	.863	.862	.860	.859	.858	23.0	19.2	16.8	21.5	18.1
100	.208	.213	.217	.217	.223	.866	.864	.862	.860	.858	22.2	21.0	18.0	22.5	18.2
200	.208	.212	.215	.215	.220	.868	.865	.864	.861	.859	25.9	22.1	22.6	26.5	24.3
500	.208	.212	.215	.215	.220	.868	.866	.865	.861	.858	31.2	24.9	24.4	29.3	22.6
∞	.207	.209	.215	.211	.217	.869	.866	.866	.863	.859	31.5	28.7	25.2	30.5	23.0

Table 37: Results with respect of HoF size for **f-mnist:7v9** dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret					al accuracy				computat	ion time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
1101 SIZE	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	тор к	Dest
0	.275	.275	.275	.274	.275	.665	.665	.665	.665	.665	81.1	81.1	81.1	81.1	81.1
10	.246	.254	.256	.264	.274	.680	.676	.676	.670	.669	91.4	84.2	81.6	87.6	79.7
20	.246	.254	.254	.262	.270	.681	.677	.676	.670	.670	95.9	91.0	97.2	97.0	81.4
50	.244	.252	.253	.262	.269	.681	.677	.677	.672	.670	116.2	102.4	100.5	104.4	91.7
100	.242	.248	.250	.258	.266	.683	.679	.678	.672	.670	125.8	110.7	114.8	123.3	90.1
200	.241	.246	.249	.255	.263	.685	.680	.680	.671	.672	132.6	124.1	139.0	132.1	102.7
500	.243	.248	.250	.256	.265	.684	.681	.679	.671	.672	154.2	144.5	173.1	163.3	104.6
∞	.239	.242	.249	.247	.261	.686	.683	.682	.678	.674	159.0	143.8	167.0	169.4	104.2

Table 38: Results with respect of HoF size for cifar 10:0v5 dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret					al accuracy				computat	ion time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Ract	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect
1101 SIZE	tree	mixed tree	trees	торк	Desi	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	тор к	
0	.320	.324	.324	.324	.325	.681	.681	.681	.681	.681	68.8	68.8	68.8	68.8	68.8
10	.295	.303	.309	.309	.318	.687	.686	.685	.683	.682	86.1	83.7	76.6	74.8	81.7
20	.296	.303	.306	.307	.314	.688	.687	.686	.684	.683	83.8	82.0	89.3	92.1	78.2
50	.292	.299	.305	.305	.313	.688	.687	.686	.685	.684	104.3	92.4	85.4	99.4	88.1
100	.290	.296	.302	.302	.310	.690	.689	.687	.686	.684	110.0	101.6	101.0	117.5	91.8
200	.289	.295	.299	.299	.305	.692	.690	.689	.686	.684	126.0	125.9	101.7	137.4	106.7
500	.288	.294	.299	.298	.305	.692	.690	.689	.687	.684	135.1	140.4	118.3	147.5	123.6
∞	.287	.290	.298	.293	.302	.693	.691	.690	.688	.685	151.8	142.6	124.4	162.6	125.6

Table 39: Results with respect of HoF size for cifar 10:0v6 dataset. ∞ means that there was no limit no limit on HoF size.

		minima	ax regret					al accuracy				computat	ion time [s]		
HoF size	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Rect	Nash mixed	Top K as	Nash single	Тор К	Best
1101 Size	tree	mixed tree	trees	торк	Dest	tree	mixed tree	trees	тор к	Dest	tree	mixed tree	trees	1	
0	.305	.318	.318	.317	.318	.652	.652	.652	.652	.652	62.7	62.7	62.7	62.7	62.7
10	.289	.297	.303	.302	.311	.659	.657	.656	.655	.654	67.5	74.9	60.4	68.0	69.0
20	.289	.297	.300	.300	.308	.659	.658	.657	.655	.655	71.7	74.3	76.1	83.4	69.5
50	.286	.293	.299	.299	.306	.660	.658	.657	.656	.655	86.6	74.9	84.6	88.4	76.9
100	.284	.290	.296	.296	.303	.661	.660	.659	.657	.655	108.6	98.3	78.0	10.3	86.0
200	.283	.288	.293	.292	.299	.663	.661	.660	.657	.656	111.4	95.0	93.1	125.3	93.6
500	.282	.288	.293	.292	.299	.663	.661	.660	.658	.655	120.0	119.8	102.6	136.5	109.2
∞	.281	.284	.292	.287	.295	.664	.662	.661	.660	.656	123.8	120.5	113.2	139.7	112.1

Table 40: Results with respect of HoF size for cifar10:4v8 dataset. ∞ means that there was no limit no limit on HoF size.